SOME BIRATIONAL GEOMETRIC ASPECTS OF MODULI SPACES OF SHEAVES ON SURFACES VIA BRIDGELAND WALL-CROSSING

by

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ABSTRACT

We study some birational geometric aspects of moduli spaces of semistable sheaves on surfaces. We observe that moduli spaces of semistable sheaves on a Del Pezzo surface are Mori-dream spaces and, following the techniques introduced by Arcara, Bertram, Coskun, and Huizenga, relate the Mori chambers to chambers in the Stability manifold introduced by Bridgeland. In the special case when $X = \mathbb{P}^2$, the wall-crossing phenomena for stability conditions (and therefore the wall-crossing in $\overline{NE}(X)$) can be analyzed very closely. In fact, for the case of 1-dimensional sheaves with vanishing Euler-Poincaré characteristic, we can describe several of the birational models, including the minimal. As we will study later, the wall-crossing phenomena in this case gives information about existence of flips of Secant varieties of Veronese surfaces. To do this we will need a generalization of a duality result of Maican for moduli spaces of 1-dimensional plane sheaves to any moduli space of Bridgeland semistable objects. In particular, we prove that Maican's result holds on arbitrary smooth complex surfaces. Finally, we show that the change of polarization for moduli spaces of sheaves on a smooth projective complex surface X, and the birational geometry of X itself, are a consequence of the wall-crossing phenomena on $\mathrm{Stab}(X)$.

To my beloved Malva

CONTENTS

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I am indebted to my advisor, Professor Aaron Bertram. I want to thank him for introducing me to the subject of Stability Conditions and for being a constant source of geometric enlightenment. Most of the mathematical intuition that I have developed during my time as a graduate student comes from discussions in his office. His encouragement and patience throughout the years were crucial in the culmination of this project.

CHAPTER 1

INTRODUCTION

In Algebraic Geometry moduli theory refers to classification problems. The question is if it is possible to find a suitable space (algebraic variety, scheme, etc.) parameterizing certain type of geometric objects (vector spaces, vector bundles, subvarieties, coherent sheaves, etc.) up to some equivalence and such that this parameter space also remembers the objects themselves and their variation in families (for some given notion of family). For instance, if one wants to construct a space parameterizing lines in $\mathbb{R}^3$ passing through the point origin, one gets the projective space $\mathbb{R}P^2$, which is the quotient $S^2/(\mathbb{Z}/2\mathbb{Z})$ of the sphere by the antipodal map.

Intuitively if M is a moduli space parameterizing certain equivalence classes of objects, then one would like for maps $\{\bullet\} \to M$ to be in correspondence with the objects we classify, and more generally, for maps $B \to M$ to be in correspondence with families of objects parametrized by B. To see that this property is not easily achieved let us look at the problem of classifying vector spaces of dimension 1 up to isomorphism. One would like for M to consist of only one point. Thus, for any space B there would be only one map $B \to M$. However, the intuitive notion for a family of lines over B is that of a line bundle, but then this would say that any space admits a single line bundle, which fails to be the case for almost every choice of B.

Moduli spaces do not always exist and even if they do, there is no reason why they should be nice spaces (irreducible, smooth, complete, projective, etc.). In this thesis we will focus on the study of the geometry of a particular well known moduli space, namely the moduli space of semistable sheaves on a complex projective surface X. When $X = \mathbb{P}^2$ and this moduli space is nonempty, it enjoys very nice properties.

There are several things we have to notice. First of all, we want to construct a parameter space that is "reasonable" so that it is as close as possible to an algebraic variety. This creates the necessity of restricting ourselves to sheaves that have some fixed topological type $v = (r, L, \chi)$, namely rank, determinant, and Euler characteristic, and that satisfy a technical condition called semistability. One of the advantages of restricting to semistable sheaves is that one can construct a scheme $\mathcal{Q}$ and a coherent sheaf $\mathcal{F}$ on $\mathcal{Q} \times X$ such that every semistable coherent sheaf of topological type v is the restriction of $\mathcal{F}$ to some fiber $\{q\} \times X$. Since in general the scheme $\mathcal{Q}$ only overparameterizes our sheaves, then one would like to take a quotient of this scheme (generally by an affine group) that serves our purposes.

The construction is as follows: since X is a complex projective variety we can fix an embedding of $X \hookrightarrow \mathbb{P}^N$. Let $\mathcal{O}_X(1)$ be the restriction to X of $\mathcal{O}_{\mathbb{P}^N}(1)$. We can find $n >> 0$, depending only on v, such that for every semistable coherent sheaf $\mathcal{F}$ of topological type v one has $H^i(X, \mathcal{F}(n)) = 0$ for $i > 0$, and the evaluation map

$$H^0(X, \mathcal{F}(n)) \otimes \mathcal{O}(-n) \to \mathcal{F}$$

is surjective. Thus, the Grothendieck's Quot scheme $\mathrm{Quot}(\mathbb{C}^{P(n)} \otimes \mathcal{O}(-n), P(n))$ parameterizing quotients of $\mathbb{C}^{P(n)} \otimes \mathcal{O}(-n)$ with Hilbert polynomial $P(n)$ is a scheme that overparametrizes our sheaves. This overparametrization is encoded by the action of $GL(P(n))$. The moduli space is the projective quotient

$$\mathrm{Quot}(\mathbb{C}^{P(n)} \otimes \mathcal{O}(-n), P(n))^{ss} // SL(P(n)).$$

In the case of curves, a torsion-free sheaf (therefore a vector bundle) E is in the open set of semistable points precisely if for every subsheaf $F \subset E$ one has the inequality of slopes

$$\mu(F) := \frac{\deg(F)}{r(F)} \leq \mu(E) = \frac{\deg(E)}{r(E)}.$$

Semistable bundles are remarkable because they generate (by extensions) the category Coh(X). Indeed, every coherent sheaf has a Harder-Narahsimhan filtration

$$0 \subset E_1 \subset \cdots E_{n-1} \subset E_n = E$$

with semistable factors $F_i = E_i/E_{i-1}$ of degreasing slopes (where for torsion sheaves declare $\mu = +\infty$). One could as well look at coherent sheaves E on a surface whose support is a curve, in this case $r(E) = 0$, but $\det(E) \neq 0$ and the stability condition becomes

$$\frac{\chi(F)}{\det(F) \cdot H} \leq \frac{\chi(E)}{\det(E) \cdot H}$$

where H is the ample class given the embedding of X. In general what happens is that the slope has to be replaced by the reduced Hilbert polynomial of the sheaf. This kind of semistability is called Gieseker semistability.

The classical way to study the birational geometry of moduli spaces of Gieseker semistable sheaves is to perturb the character chosen in the GIT construction. When X is a curve, this is the work of Bertram [Ber97] and Thaddeus [Tha94].

The way stability conditions come to play is to define a general notion of stability not only for sheaves but for elements in some abelian subcategory of the derived category $D^b(\mathrm{Coh(X)})$. This includes Gieseker stability. For example, in the case of sheaves supported on curves one can define stability conditions on different abelian subcategories that include all torsion sheaves, then we have to test stability with more objects which implies that a Gieseker semistable sheaf has chance to become unstable and new objects (complexes) can become stable.

Roughly speaking, a stability condition consists of two kinds of data: an abelian subcategory $\mathcal{A} \subset D^b(X)$ and a linear function $Z : K(\mathcal{A}) \to \mathbb{C}$ that behaves as $-\deg + \sqrt{-1}\mathrm{rk}$ does for curves, i.e., the imaginary part $\Im(Z) \geq 0$, and if $\Im(Z(E)) = 0$, then the real part satisfies $\Re(Z(E)) < 0$. One also requires for objects of $\mathcal{A}$ to have Harder-Narasimhan filtrations in $\mathcal{A}$ with respect to the slope

$$\mu_Z = -\frac{\Re(Z)}{\Im(Z)}.$$

Stability conditions on triangulated categories were introduced by Bridgeland in [Bri07], who also constructed the first family of nontrivial examples for $K3$ surfaces in [Bri08]. In [AB13] Arcara and Bertram extended these examples for an arbitrary smooth projective surface. [AB13] also provides us with a new perspective. By studying the variation of a particular family of stability conditions on a simple $K3$ surface (X, H) for a particular topological type, the authors construct a sequence of birational transformations for the blow-up of the complete linear series $|H|$ along X, flipping the "secant" varieties of X.

The wall-crossing phenomena was studied in [ABCH13] in detail for the Hilbert scheme of points on $\mathbb{P}^2$, where it was indicated that varying the family of stability conditions introduced in [AB13] for the topological type $(1, 0, -n)$ corresponds to running a directed MMP on $Hilb^n(\mathbb{P}^2)$. As we will see in Chapter 2, this is the case for every primitive topological type. More precisely, we have

Theorem (Theorem 3.6)**.** *Let $X = \mathbb{P}^2$ polarized by the hyperplane class H and $M_H(v)$ be the coarse moduli space parametrizing S-equivalence classes of Gieseker semistable sheaves of primitive topological type $v = (r, c_1, ch_2)$.*

(a) When $r \neq 0$ and $M_H(v)$ is nonempty, then it is a smooth weak Fano variety (i.e., $-K_{M_H(v)}$ is big and nef) of Picard rank at most 2. In particular, it is a Mori dream space.

(b) Starting from $t \gg 0$ and decreasing t corresponds to running a directed MMP on $M_H(v)$. As long as the generic point of the exceptional loci of each contraction is a sheaf, each birational model (both in the interior of the Mori chamber and at the wall) we get in the directed MMP is isomorphic to the normalization of the main component of the Bridgeland moduli in the corresponding Bridgeland chamber and wall.

In [Mai10] M. Maican proves that the map $\mathcal{F} \mapsto \mathscr{E}xt^{n-1}(\mathcal{F}, \omega_{\mathbb{P}^n})$ induces an isomorphism between the moduli spaces $N_{\mathbb{P}^n}(r, \chi)$ and $N_{\mathbb{P}^n}(r, -\chi)$ of Gieseker semistable sheaves with Hilbert polynomials $P = rm + \chi$ and $P^D = rm - \chi$, respectively. The moduli spaces $N_X(r, \chi)$ were constructed by C. Simpson [Sim94] for any smooth projective surface via Geometric Invariant Theory (GIT), and they were proven to be projective. After identifying $N_X(r, \chi)$ with a moduli space of Bridgeland semistable objects on X, Maican's theorem can be obtained as a particular case of the following

Theorem (Theorem 4.4)**.** *The functor $(\,\cdot\,)^D := R\mathcal{H}om(\cdot, \omega_X)[1]$ induces an isomorphism between the moduli spaces $M_{D,tH}(v)$ and $M_{-D+K_X,tH}(v^D)$ of Bridgeland semistable* *objects, provided these moduli spaces exist and $Z_{D,tH}(v)$ belongs to the open upper half plane.*

The following Corollary was obtained by Saccà in her thesis [Sac13].

Corollary (Corollary 4.7)**.** *There is an isomorphism $N_X([C],\chi) \cong N_X([C],-\chi)$ mapping the S-equivalence class of a sheaf $\mathcal{F}$ to the S-equivalence class of $\mathscr{E}xt^1(\mathcal{F},\omega_X)$.*

In a latter paper [Mai13], Maican constructs cohomological stratifications of the Gieseker moduli $N_{\mathbb{P}^2}(6,\chi)$. Using those strata we can get exceptional loci for birational transformations of $N_{\mathbb{P}^2}(6,\chi)$, as it was done in [BMW14] for $N_{\mathbb{P}^2}(4,2)$ and $N_{\mathbb{P}^2}(5,0)$. However, there is no bijective correspondence between the cohomological strata and the Bridgeland walls. Indeed, it was shown in [CC13] that in the case of $N_{\mathbb{P}^2}(6,1)$ a cohomological strata may be the object of several contractions when running the MMP, giving rise to several Bridgeland walls. Nevertheless, when $\chi = 0$ we can identify all rank-1 walls even when Maican-type stratifications are unknown. In this case, by restricting the Bridgeland wall-crossing on a suitable subvariety of a model of $N_{\mathbb{P}^2}(d,0)$ (d odd), and following the spirit of [AB13], we construct a sequence of flips for the blow-up of the linear series $|\mathcal{O}(d-3)|$ along the Veronese surface. The first of these flips coincides with the one constructed by Vermeire in [Ver01].

It is important to emphasize that the notion of Gieseker semistability depends on the choice of an embedding $X \hookrightarrow \mathbb{P}^N$ (i.e., the choice of an ample class on X). It was established in the 90s that the ample cone of the surface has a wall and chamber decomposition satisfying that $M_H(v)$ and $M_{H'}(v)$ are isomorphic when H and H' belong to the same chamber. Results obtained independently by Ellingsrud and Göttsche [EG95] and Friedman and Qin [FQ95] for rank-two sheaves, and by Matsuki and Wentworth [MW97] in arbitrary positive rank via variation of GIT, show that when crossing a wall in the ample cone of the surface, the moduli space $M_H(v)$ goes through a sequence of "Thaddeus flips" of moduli spaces of "twisted" sheaves. In Chapter 5, we will show that this result is a consequence of the wall-crossing phenomena for Bridgeland stability conditions. We obtain the following

Theorem (Theorem 5.7)**.** *Given H' and H'' two ample classes in adjacent chambers in the wall and chamber decomposition of $Amp(X)$ for the class v, then there is a one dimensional family of stability conditions $\{\sigma_t\}_{t\in(-1,1)}$ and rational numbers $-1 = t_0 < t_1 < \cdots < t_n = 1$ such that each moduli space $M_{\sigma_t}(v)$ of Bridgeland semistable objects is a moduli space of twisted sheaves for every t and is constant on (t_i, t_{i+1}). It equals $M_{H'}(v)$ for $t \in (t_0, t_1)$ and equals $M_{H''}(v)$ for $t \in (t_{n-1}, t_n)$.*

In particular, by identifying a surface with its Hilbert scheme of length-one subschemes, we can see the classical birational geometry of complex surfaces as a consequence of the wall-crossing phenomena:

Theorem (Theorem 5.10)**.** *[Tod12, Corollary 1.4] Let X be a smooth projective complex surface and let $\pi\colon X \to Y$ be the blow down of a -1-curve $C \subset X$. Then there is a continuous one parameter family of Bridgeland stability conditions $\{\sigma_t\}_{t\in(-1,1)}$ on $D^bCoh(X)$ such that $M_{\sigma_t}([\mathcal{I}_p])$ is isomorphic to X for $t > 0$ and isomorphic to Y for $t < 0$.*

Other than specified we will use the following standard notation:

- $\Re(z)$, $\Im(z)$ denote the real and imaginary parts of the complex number z.
- For an abelian category $\mathcal{A}$, we denote by $K(\mathcal{A})$, its Grothendieck group.
- $D^b(X)$ is the bounded derived category of X.
- We use $\mathcal{H}^i(\cdot)$ to denote the cohomology sheaves of an object in the derived category and $H^i(\cdot)$ for the cohomology groups of a sheaf.
- For a smooth projective surface X, the topological type $v \in \mathbb{Z} \oplus NS(X) \oplus \frac{1}{2}\mathbb{Z}$ of an object $E \in D^b(X)$ is its Chern character vector.
- $M_H(v)$ denotes the moduli space of Gieseker semistable sheaves of topological type v with respect to the polarization $H \in \operatorname{Pic} X$.
- $M_{s,t}(v)$ denotes the Bridgeland moduli space of $\sigma_{s,t}$-semistable objects of topological type v.
- We refer to an object F fitting into an exact sequence $A \hookrightarrow F \twoheadrightarrow B$ as an extension.

CHAPTER 2

PRELIMINARIES

For the first-time reader, the amount of definitions and technicalities in this chapter may be overwhelming. The reader should be aware that this chapter contains very few proofs, only those corresponding to results whose proofs in the generality needed in this thesis are not covered in the standard references. This chapter is a collection of definitions and, by this time, well known results in the area of Stability Conditions; these are the basis on which we build the results of the following chapters. Before we start formally, here is a summary of the main ideas you will find in the coming sections:

- Let (X, H) be a polarized complex projective surface (this is equivalent to choose an embedding of X in a projective space). There exists an abelian subcategory $\mathcal{A} \subset D^b(X)$ (depending on H) whose elements are two-term complexes satisfying some technical condition.

- Short exact sequences in $\mathcal{A}$ are distinguished triangles in $D^b(X)$ with vertices in $\mathcal{A}$. Moreover, the triangle

$$\mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E)$$

  is a short exact sequence in $\mathcal{A}$ for every $E \in \mathcal{A}$.

- For every real number $t > 0$ one can define linear functions $d_t, r_t \colon K(\mathcal{A}) \to \mathbb{R}$ satisfying

$$r_t(E) = 0 \Rightarrow d_t(E) > 0 \text{ for all } E \in \mathcal{A}.$$

  We call d_t and r_t the Bridgeland degree and rank in analogy with the usual degree and rank for coherent sheaves on curves.

- The ratio $\mu_t = d_t/r_t$ is called the Bridgeland slope. We say that an object $E \in \mathcal{A}$ is (semi)stable if for every subobject $F \hookrightarrow E$ one has $\mu_t(F)(\leq) < \mu_t(E)$.

- Every element $E \in \mathcal{A}$ has a unique filtration in $\mathcal{A}$ (its Harder-Narasimhan filtration) of the form

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots E_{n-1} \hookrightarrow E_n = E$$

  whose factors $F_i = E_i/E_{i-1}$ are μ_t-semistable with $\mu_t(F_1) > \mu_t(F_2) > \cdots > \mu_t(F_n)$.

- Every μ_t-semistable $E \in \mathcal{A}$ has a finite Jordan-Holder filtration, i.e., a filtration whose factors are μ_t-stable of the same slope $\mu_t(E)$. Two μ_t-semistable objects of $\mathcal{A}$ are said to be S-equivalent if they have isomorphic stable factors (up to a permutation).

- If $X = \mathbb{P}^2$, then for every polarization H and every $t > 0$ there are projective coarse moduli spaces $M_t^H(v)$ parametrizing S-equivalence classes of families of μ_t-semistable objects of Chern character v.

- For $t \gg 0$ the moduli space $M_t^H(v)$ coincides with the classical moduli space of Gieseker semistable sheaves of Chern character v.

- There are only finitely many isomorphism classes of spaces $M_t^H(v)$, and decreasing t corresponds to run a directed minimal model program on the Gieseker moduli space.

2.1 Stability conditions

Let X be a smooth projective complex variety, and denote by $D^b(X)$ its bounded derived category. $D^b(X)$ is roughly speaking the category of complexes of sheaves where quasi-isomorphisms are formally inverted. This, for example, allows us to identify a coherent sheaf with the complex formed by any of its free resolutions. A key point here is that the category $D^b(X)$ is a triangulated category.

Even though the derived category $D^b(X)$ is not abelian (unless the category CohX is semisimple), it has nice abelian subcategories.

Definition 2.1. A t-structure on a triangulated category $\mathcal{D}$ is a pair of strictly full subcategories $(D^{\leq 0}, D^{\geq 0})$ satisfying:

(a) $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 0} \subset D^{\geq 1}$.

(b) $\mathrm{Hom}(X, Y) = 0$ for all $X \in D^{\leq 0}$, $Y \in D^{\geq 1}$.

(c) For all $E \in \mathcal{D}$ there is a distinguished triangle $A \to E \to B \to A[1]$ with $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

Where $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$. The heart of the t-structure is $\mathcal{A} = D^{\geq 0} \cap D^{\leq 0}$.

Theorem 2.2 (See [GM03]). *$\mathcal{A} = D^{\geq 0} \cap D^{\leq 0}$ is an abelian category with short exact sequences being the triangles in $\mathcal{D}$ with vertices in $\mathcal{A}$.*

In fact, it follows that if $(D^{\leq 0}, D^{\geq 0})$ is the heart of a t-structure on $\mathcal{D}$, then

$$D^{\geq 1} = (D^{\leq 0})^{\perp} = \{Y \in \mathcal{D} \colon \mathrm{Hom}(X, Y) = 0 \ \text{ for all } \ X \in D^{\leq 0}\}.$$

Set $F = D^{\leq 0}$ then $\mathcal{A} = F \cap F^{\perp}[1] \subset \mathcal{D}$.

Definition 2.3. A t-structure $F \subset \mathcal{D}$ is bounded if

$$\mathcal{D} = \bigcup_{i,j} F[i] \cap F^{\perp}[j].$$

It is worth noticing that the bounded t-structure F is determined by its heart. Indeed, F is the extension closure of the subcategories $\mathcal{A}[i]$ with $i \geq 0$. Thus the following lemma is an immediate consequence of the definitions:

Lemma 2.4. *If $\mathcal{A} \subset \mathcal{D}$ is a full additive subcategory, then $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ if and only if the two conditions hold:*

(a) If A and B are objects of $\mathcal{A}$ then $\mathrm{Hom}_{\mathcal{D}}(A, B[k]) = 0$ for $k < 0$,

(b) For every non-zero object $E \in \mathcal{D}$, there are integers $m < n$ and a collection of triangles

$$\begin{tikzcd}[column sep=small]
0 = E_m \arrow[rr] & & E_{m+1} \arrow[rr] \arrow[dl] & & E_{m+2} \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_{m+1} \arrow[ul] & & A_{m+2} \arrow[ul] & & & & A_n \arrow[ul] &
\end{tikzcd}$$

with $A_i[i] \in \mathcal{A}$ for all i.

Definition 2.5. A prestability condition on X is a pair $(Z, \mathcal{A})$ consisting of a linear function $Z : K(X) \to \mathbb{C}$ called the central charge and the heart $\mathcal{A}$ of a bounded t-structure on $D^b(X)$, such that:

(a) $\Im(Z(E)) \geq 0$ for all $E \in \mathcal{A}$ and

(b) If $\Im(Z(E)) = 0$ and $E \neq 0$ then $\Re(Z(E)) < 0$.

For every prestability condition, one can define a slope function

$$\mu_Z = \frac{-\Re(Z)}{\Im(Z)},$$

which gives us a notion of (semi)stability: an object E in the category $\mathcal{A}$ is said to be Z-(semi)stable if for any inclusion $A \hookrightarrow E$ of objects in $\mathcal{A}$ one has

$$\mu_Z(A)(\leq) < \mu_Z(E).$$

Definition 2.6. A prestability condition $(Z, \mathcal{A})$ is a stability condition if it has the Harder-Narasimhan property:

- (HN) Every non-zero object $E \in \mathcal{A}$ admits a finite filtration in $\mathcal{A}$

$$0 \subset E_0 \subset E_1 \subset \cdots \subset E_n = E$$

uniquely determined by the property that each quotient $F_i := E_i/E_{i-1}$ is Z-semistable, and $\mu_Z(F_1) > \mu_Z(F_2) > \cdots > \mu_Z(F_{n-1})$.

Example 2.6.1. If $X = C$ is a smooth projective complex curve, then ordinary degree and rank of coherent sheaves give a stability condition on $\mathcal{A} = D^b(\mathrm{CohC})$:

$$Z(\mathcal{F}) = -\deg(\mathcal{F}) + \sqrt{-1}\mathrm{rk}(\mathcal{F}).$$

However, when X is a surface this is not the case. One can still define a Mumford slope (with respect to some polarization H):

$$\mu_H(E) = \frac{c_1(E) \cdot H}{\mathrm{rk(E)}},$$

but this does not come from any stability condition on CohX since $c_1(\mathbb{C}_p) = 0$ and $\mathrm{rk}(\mathbb{C}_\mathrm{p}) = 0$. However, it is true that every coherent sheaf E has a filtration

$$E_0 \subset \cdots \subset E_n = E$$

such that E_0 is the torsion subsheaf of E and for every $i > 0$, the factors E_i/E_{i-1} are semistable of degreasing slopes. If $\mathrm{rk(E)} > 0$, we set $\mu_H^+(E) = \mu_H(E_1/E_0)$ and $\mu_H^-(E) = \mu_H(E/E_{n-1})$. For torsion sheaves we declare $\mu_H(E) := +\infty$. $\square$

Let $\sigma = (Z, \mathcal{A})$ be a stability condition on X. For any non-zero object $E \in \mathcal{A}$ one can write $Z(E) = |Z(E)|e^{\pi\sqrt{-1}\phi}$ for a unique $\phi \in (0,1]$. We say that E has phase ϕ. For every $\phi \in (0,1]$ we denote by $\mathcal{P}(\phi)$ the subcategory consisting of σ-semistable objects of phase ϕ. Inductively one can define $\mathcal{P}(\phi+1) := \mathcal{P}(\phi)[1]$. Notice that the categories $\mathcal{P}(\phi)$ are abelian with simple objects being the Z-stable objects of phase ϕ. For a bounded interval $I \subset \mathbb{R}$ we denote $\mathcal{P}(I)$ the subcategory extension-generated by σ-semistable objects of phase in the interval I. Here by extension we mean a short exact sequence of objects in $\mathcal{A}$. For instance, since every object in $\mathcal{A}$ has a Harder-Narasimhan filtration, then every element of $\mathcal{A}$ is in the extension closure of its semistable factors and thus $\mathcal{P}(0,1] = \mathcal{A}$.

One can define (semi)stability in terms of phase just by declaring an object E to be (semi)stable if every subobject has (smaller)strictly smaller phase. This is equivalent to the definition using slopes since for an object $E \in \mathcal{A}$ of phase ϕ one has

$$\mu_Z(E) = -\cot(\pi\phi).$$

Let $\sigma = (Z, \mathcal{A})$ be a stability condition. If $E \in \mathcal{D}$ has a Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

with σ-semistable factors $F_i = E_i/E_{i-1}$, then we define $\phi_\sigma^+(E) := \phi(F_1)$, $\phi_\sigma^-(E) := \phi(F_n)$, and $m_\sigma(E) := \sum |Z(F_i)|$.

An easy but important consequence of the definition of stability is

Proposition 2.7 (Schur's lemma)**.** *Let $\sigma = (Z, \mathcal{A})$ be a stability condition.*

(a) If E is σ-stable then $\mathrm{Hom(E, E)} = \mathbb{C}$.

(b) If A, B are different σ-stable objects of the same phase, then $\mathrm{Hom(A, B)} = 0$.

(c) If $A \in \mathcal{P}(\phi_1)$, $B \in \mathcal{P}(\phi_2)$ with $\phi_1 > \phi_2$, then $\mathrm{Hom(A, B)} = 0$.

Remark 2.8. If $\sigma = (Z, \mathcal{A})$ is a stability condition then it follows that if $A, B \in \mathcal{A}$ are two objects such that $\phi^-_\sigma(A) > \phi^+_\sigma(B)$, then $\mathrm{Hom(A, B)} = 0$. The proof of this fact is a simple two-step induction on the number of semistable factors of B and A.

Let $E \in \mathcal{P}(\phi)$. A finite Jordan-Holder filtration of E is a chain

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

such that the factors

$$F_i := E_i / E_{i-1} \in \mathcal{P}(\phi)$$

are stable.

Definition 2.9. A stability condition is called locally finite if there is some $\delta > 0$ such that each quasi-abelian category $\mathcal{P}(\phi - \delta, \phi + \delta)$ is of finite length. For a locally finite stability condition the categories $\mathcal{P}(\phi)$ have finite length. In particular, every semistable object has a finite decomposition series, i.e., a finite Jordan-Holder filtration.

Definition 2.10. Let σ be a locally finite stability condition. Two objects $A, B \in \mathcal{P}(\phi)$ are called S-equivalent if they have isomorphic stable factors.

Definition 2.11. [AP06]. Let S be a scheme of finite type over $\mathbb{C}$. A flat family of objects in $\mathcal{A}$ parametrized by S is an object $E \in D^b(X \times S)$ such that for every closed point $s \in S$ we have

$$Li_s^*(E) \in \mathcal{A},$$

where $i_s : X \to X \times S$ is the inclusion at $s \in S$.

2.2 Geometric stability conditions on surfaces

As we explained before, the standard rank and degree of a coherent sheaf do not define a stability condition on any surface. A large class of examples of stability conditions on surfaces were constructed by Bridgeland [Bri08] in the case of $K3$ surfaces and generalized by Arcara-Bertram [AB13] for any smooth projective surface. The idea is to define nice abelian subcategories of $D^b(X)$ where some generalized rank and degree functions form good stability functions giving actual stability conditions. Fix a very ample line bundle $\omega \in Pic(X)$. One defines, for every $s \in \mathbb{R}$, a pair of subcategories of Coh(X):

$$\begin{aligned} \mathcal{Q}_s &= \{E \colon E \text{ is torsion sheaf or } \mu_\omega^-(E) > s\}, \\ \mathcal{F}_s &= \{E \colon E \text{ is torsion-free sheaf and } \mu_\omega^+(E) \leq s\}. \end{aligned}$$

The subcategories $\mathcal{Q}_s, \mathcal{F}_s$ are full, and $(\mathcal{Q}_s, \mathcal{F}_s)$ is a torsion pair, i.e.,

- $\mathrm{Hom}(Q, F) = 0$ for all $Q \in \mathcal{Q}_s$, $F \in \mathcal{F}_s$.
- Every coherent sheaf E fits into an exact sequence

  $$0 \to Q \to E \to F \to 0$$

  for some $Q \in \mathcal{Q}_s$, $F \in \mathcal{F}_s$ (see Example 2.6.1). This short exact sequence is unique up to isomorphisms of extensions.

By general theory of torsion pairs we know that the extension closure $\langle \mathcal{Q}_s, \mathcal{F}_s[1] \rangle$ is the heart of a bounded t-structure, more precisely it is the full subcategory

$$\mathcal{A}_s = \{E \in D^b(X) \colon H^{-1}(E) \in \mathcal{F}_s,\ H^0(E) \in \mathcal{Q}_s, \text{ and } H^i(E) = 0 \text{ for } i \neq -1, 0\}.$$

Theorem 2.12 ([AB13],[Bri08]). *Let $\beta, \omega \in \mathrm{Num}(X)_{\mathbb{R}}$, ω ample. Then for any $t > 0$*

$$Z_{\beta, t\omega}(E) = -\int_X e^{-\beta - \sqrt{-1} t\omega} ch(E)$$

is the charge of a locally finite stability condition on $\mathcal{A}_{\beta\omega}$.

Definition 2.13. A stability condition $\sigma = (Z, \mathcal{A})$ is called geometric if all skyscraper sheaves $\mathbb{C}_x$ are stable of the same phase.

By the definition of $\mathcal{A}_{\beta\omega}$ one knows that $\mathbb{C}_x \in \mathcal{A}_{\beta\omega}$ for all $x \in X$. Moreover, $\mathbb{C}_x$ does not have other subobjects in $\mathcal{A}_{\beta\omega}$ other than 0 and $\mathbb{C}_x$. Thus the stability conditions $\sigma_{\beta, t\omega} = (Z_{\beta, t\omega}, \mathcal{A}_{\beta\omega})$ are geometric. Bridgeland also characterizes the geometric stability conditions up to the action of $\mathbb{C}^*$:

Lemma 2.14. *[Bri08, Lemma 10.1] Assume that $\sigma = (Z, \mathcal{A})$ is a stability condition such that every skyscraper sheaf $\mathbb{C}_x$ is stable with $Z(\mathbb{C}_x) = -1$. Let $E \in D^b(X)$. Then*

(a) *if $E \in \mathcal{P}(0,1]$, then $\mathcal{H}^i(E) = 0$ unless $i = 0, -1$,*

(b) *If E is stable, then either $E = \mathbb{C}_x$ for some $x \in X$, or $E = F[1]$ for some locally free sheaf $F \in \mathrm{Coh(X)}$,*

(c) *if $E \in \mathrm{Coh(X)}$, then $E \in \mathcal{P}(-1,1]$; if E is torsion, then $E \in \mathcal{P}(0,1]$,*

(d) *the pair of subcategories*

$$\mathcal{T} = \mathrm{Coh(X)} \cap \mathcal{P}(0,1], \;\; \mathcal{F} = \mathrm{Coh(X)} \cap \mathcal{P}(-1,0]$$

define a torsion pair on $\mathrm{Coh(X)}$ and $\mathcal{P}(0,1]$ is the corresponding tilt.

The proof of part (a) in the lemma above relies on a result of Bridgeland and Maciocia. We state it here for future references

Proposition 2.15. *[BM02, Prop. 5.4] Let X be a quasi-projective scheme and let $E \in D^b(X)$ be a non-zero object. Assume that there is an integer $k \geq 0$ such that for all points $x \in X$*

$$\mathrm{Hom}^i(E, \mathbb{C}_x) = 0 \;\; \textit{unless} \;\; 0 \leq i \leq k.$$

Then E is quasi-isomorphic to a complex of locally free sheaves of the form

$$0 \to L_{-k} \to L_{-k+1} \to \cdots \to L_1 \to L_0 \to 0.$$

Remark 2.16. Proposition 2.15 constrains the shape of a stable object. Assume that $(Z, \mathcal{A})$ is a geometric stability condition satisfying $Z(\mathbb{C}_x) = -1$, and let $E \in \mathcal{A}$ be a Z-stable object of phase in $(0,1)$. Since $\mathbb{C}_x$ is stable of phase 1, then

$$\begin{aligned} \mathrm{Hom}^i(E, \mathbb{C}_x) &= \mathrm{Hom(E, \mathbb{C}_x[i])} = 0 \;\text{ for }\; \mathrm{i} < 0, \text{ and} \\ \mathrm{Hom}^i(E, \mathbb{C}_x) &= \mathrm{Hom}^{2-i}(\mathbb{C}_x, E \otimes^L \omega_X)^\vee = \mathrm{Hom}^{2-i}(\mathbb{C}_x, E)^\vee = 0 \;\text{ for }\; i \geq 2. \end{aligned}$$

Thus, by Proposition 2.15, E is quasi-isomorphic to a two-term complex

$$E^{-1} \to E^0$$

with E^{-1} and E_0 locally free. □

2.3 Stability conditions at the large volume limit

As before, let X denote a smooth projective complex surface, $L, H \in Pic(X) \otimes \mathbb{Q}$ with H ample. For a torsion-free coherent sheaf $\mathcal{E} \in \mathrm{Coh(X)}$, define the slope functions

$$\mu_H(\mathcal{E}) = \frac{c_1(\mathcal{E})}{r(\mathcal{E})} \cdot H, \qquad \nu_L(\mathcal{E}) = \frac{\chi(\mathcal{E}) - c_1(\mathcal{E}) \cdot L}{r(\mathcal{E})}.$$

Definition 2.17. [[MW97]] A torsion-free coherent sheaf $\mathcal{E}$ is said to be L-twisted H-Gieseker semistable if for any subsheaf $A \subset \mathcal{E}$ we have

$$\mu_H(A) < \mu_H(\mathcal{E}) \quad \text{or} \quad (\mu_H(A) = \mu_H(\mathcal{E}) \quad \text{and} \quad \nu_L(A) \leq \nu_L(\mathcal{E}))\,.$$

For a fixed topological type v with $r(v) > 0$, denote by $M_{H,L}(v)$ the moduli space of L-twisted semistable sheaves. This space was constructed by Matsuki and Wentworth [MW97] when studying the effect on $M_H(v)$ of changing the polarization H. It has been proven in [ABCH13] that when $X = \mathbb{P}^2$, then $\sigma_{sH,tH}$-stability and Gieseker stability coincide for $t \gg 0$. This was also considered by Bridgeland [Bri08] when he proved that for the case of $K3$ surfaces the limit of $\sigma_{D,tH}$-(semi)stability is D-twisted H-Gieseker (semi)stability. We claim that Bridgeland's idea generalizes to any surface and moreover

Theorem 2.18. *If $D = \dfrac{K_X}{2} + L$ then there exists $t_0 >> 0$ such that $\sigma_{D,tH}$-(semi)stability coincides with L-twisted H-Gieseker (semi)stability for all $t \geq t_0$.*

This theorem is a corollary of [Bri08, Proposition 10.2], whose content we split into the following two propositions:

Proposition 2.19. *If an element $E \in \mathcal{A}_{DH}$ of fixed topological type $v = (r, c_1, ch_2)$ ($r > 0$ and $\frac{c_1}{r} > DH$) is $\sigma_{D,tH}$-semistable for $t \gg 0$, then E is an L-twisted H-Gieseker semistable sheaf.*

Proposition 2.20. *Let E be an L-twisted H-Gieseker semistable sheaf and let $A \hookrightarrow E$ be an inclusion of objects in the category $\mathcal{A}_{DH}$. Then there exists $t_0 > 0$ such that $\mu_{D,tH}(A) < \mu_{D,tH}(E)$ for $t \geq t_0$.*

Proof. Again v is chosen such that $r > 0$ and $\frac{c_1}{r} > DH$ so that $E \in \mathcal{A}_{DH}$.

Assume that there is a short exact sequence in $\mathcal{A}_{DH}$.

$$0 \to A \to E \to B \to 0.$$

Notice that A must be a sheaf since there is an inclusion of sheaves $\mathcal{H}^{-1}(A) \hookrightarrow \mathcal{H}^{-1}(E) = 0$; however, A may not be a subsheaf of E. Nevertheless, by using the (twisted) semistability of E one sees that $\mu_H(A) \leq \mu_H(E)$, and that if $\mu_H(A) = \mu_H(E)$, then $\nu_L(A) \leq \nu_L(E)$. If we write

$$w_t = (\nu_L(A) - \nu_L(E)) + \sqrt{-1}t(\mu_H(E) - \mu_H(A))$$

then one has

$$\frac{Z_t(E)}{r(E)} = \frac{Z_t(A)}{r(A)} + w_t.$$

If $\mu_H(A) = \mu_H(E)$, then $\operatorname{Re}(w_t) \leq 0$, and so $\phi_t(A) \leq \phi_t(E)$ for all t. Then we can assume E μ_H-stable.

We need the following boundedness result (see [Bri08, Lemma 14.3] for a proof in the K3 case):

Lemma 2.21. *Let $E \in \mathcal{A}_{DH}$ be a L-twisted H-Gieseker semistable torsion-free sheaf. Then the set of values $\nu_L(A)$ as A runs on all non-zero subobjects of E in $\mathcal{A}_{DH}$ is bounded above.*

Proof. Consider the Harder-Narasimhan filtration of A (with respect to μ_H)

$$0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset A_n = A.$$

Denote by F_i the corresponding semistable factors. Since $A \in \mathcal{A}_{DH}$, then $\mu_H(F_i) > DH$ for all i. Now, $F_1 = A_1 \subset E$, and so $\mu_H(F_1) \leq \mu_H(E)$, which implies

$$D \cdot H < \mu_H(F_i) \leq \mu_H(E) \quad \text{for all} \quad i.$$

We can write

$$\mu_H(F_i) - D \cdot H = aH^2, \quad a > 0,$$

and by the Hodge index theorem

$$-\frac{c_1(F_i)}{r(F_i)}\left(\frac{K_X}{2} + L\right) \leq a\frac{c_1(F_i)}{r(F_i)}H - \frac{c_1(F_i)^2}{2r(F_i)^2} - \frac{(D+aH)^2}{2}.$$

This gives

$$\begin{aligned}
\nu_L(F_i) &= \frac{ch_2(F_i)}{r(F_i)} - \frac{c_1(F_i)}{2r(F_i)}K_X + \chi - \frac{c_1(F_i)}{r(F_i)}L \\
&\leq \frac{ch_2(F_i)}{r(F_i)} + a\frac{c_1(F_i)}{r(F_i)}H - \frac{c_1(F_i)^2}{2r(F_i)^2} - \frac{(D+aH)^2}{2} + \chi \\
&\leq a(\mu_H(F_i) - D\cdot H) - \frac{D^2}{2} - a^2\frac{H^2}{2} + \chi \\
&\leq \frac{(\mu_H(E) - D\cdot H)^2}{H^2} - \frac{D^2}{2} + \chi.
\end{aligned}$$

Thus, there is a constant M, depending only on the numerical data of E, such that $\nu_L(F_i) \leq M$ for all semistable factors F_i of A. An inductive argument then proves that $\nu_L(A) \leq M$. □

The lemma shows that

$$\frac{\mathrm{Re}(Z_t(A))}{r(A)} > t^2\frac{H^2}{2} - M - \frac{D^2}{2}.$$

Thus, we can assume $\frac{\mathrm{Re}(Z_t(A))}{r(A)} > 0$ and consequently that $\mathrm{Re}(w_t) > 0$. We want to find t_0 such that

$$\frac{\mathrm{Im}(w_t)}{\mathrm{Re}(w_t)} > \frac{\mathrm{Im}(Z_t(A))}{\mathrm{Re}(Z_t(A))} \quad \text{for all} \quad t \geq t_0.$$

Since we have

$$\frac{\mathrm{Im}(w_t)}{\mathrm{Re}(w_t)} > \frac{t(\mu_H(E) - \mu_H(A))}{M - \nu_L(E)} \quad \text{and} \quad \frac{t(\mu_H(A) - DH)}{t^2\frac{H^2}{2} - (\frac{D^2}{2} + M)} > \frac{\mathrm{Im}(Z_t(A))}{\mathrm{Re}(Z_t(A))}$$

then such t_0 exists. □

The conclusion of Theorem 2.18 is obtained by using the following boundedness result due to Lo and Qin:

Theorem 2.22. *[LQ11, Theorem 4.5] Fix a topological type v and $\omega, \beta \in \mathrm{Num}(X)_{\mathbb{Q}}$ with ω ample. Then*

(a) the set of walls of type v intersecting the ray $\{\sigma_{\beta,t\omega}\}_{t>0}$ is locally finite,

(b) there exists $t_0 > 0$ such that no walls of type v intersect the ray $\{\sigma_{\beta,t\omega}\}_{t>0}$ for $t > t_0$.

Remark 2.23. Except for the proof of Lemma 2.21, the proof of Proposition 2.20 is exactly as it appears in Bridgeland's paper.

Remark 2.24. In Chapter 5 we will see that there is a more natural way to identify the moduli spaces of (twisted) Gieseker semistable sheaves as moduli spaces of Bridgeland semistable objects. This will require a stronger boundedness result due to Maciocia.

2.4 Wall and chamber structure

What is remarkable about the set of stability conditions is that it has the structure of a complex manifold. The connected component consisting of stability conditions satisfying the support property has a well-behaved wall and chamber structure. Fix once and for all a surjective homomorphism $v : K(\mathcal{A}) \to \Gamma$ to a finite rank lattice Γ.

Definition 2.25. Let $\sigma = (Z, \mathcal{A})$ be a stability condition whose central charge Z factors through v. We say that σ satisfies the support property if there exists a quadratic form Q on $\Gamma \otimes \mathbb{R}$ such that

- $\ker(Z)$ is negative definite with respect to Q, and
- for all σ-semistable object $E \in \mathcal{A}$, we have

$$Q(v(E)) \geq 0.$$

Let X be a smooth complex projective variety, and let $\mathcal{D} = D^b(X)$ denote its bounded derived category. We denote by $\mathrm{Stab(X)}^*$ the set of locally finite stability conditions whose central charge factors through the Chern character $ch : D^b(X) \to \Gamma := \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \frac{1}{2}\mathbb{Z}$ and that satisfy the support property. $\mathrm{Stab(X)}^*$ has a natural topology given in the following proposition.

Proposition 2.26. *[Bri07, Proposition 8.1] The function*

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^-_{\sigma_1}(E) - \phi^-_{\sigma_2}(E)|,\ |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)|,\ \left|\log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)}\right| \right\} \in [0, \infty],$$

defines a generalized metric on $\mathrm{Stab(X)}^*$.

The following is a special case (enough for our purposes) of Bridgeland's deformation result.

Theorem 2.27. *[Bri07, Theorem 1.2] Let $\mathcal{Z} \colon \mathrm{Stab}(\mathrm{X})^* \to Hom_{\mathbb{Z}}(\Gamma, \mathbb{C})$ be the map sending a stability condition $\sigma = (Z, \mathcal{A})$ to its central charge Z. Then $\mathcal{Z}$ is a local homeomorphism. Thus,* $\mathrm{Stab}(\mathrm{X})^*$ *is a complex manifold locally modeled by $Hom_{\mathbb{Z}}(\Gamma, \mathbb{C})$.*

Definition 2.28. A set of objects $S \subset \mathcal{D}$ has bounded mass in a connected component of $\mathrm{Stab}(\mathrm{X})^*$ if

$$\sup\{m_\sigma(E) \colon E \in S\} < \infty$$

for some point (and therefore for all) σ in that connected component.

Proposition 2.29. *[Bri08, Proposition 9.3] Let $S \subset \mathcal{D}$ of bounded mass in a connected component $V \subset \mathrm{Stab}(\mathrm{X})^*$, and let $K \subset V$ be a compact subset. Then there is a finite collection $\{W_\gamma \colon \gamma \in F\}$ of (real) codimension 1 submanifolds of V (not necessarily closed), such that any connected component*

$$C \subset K \setminus \bigcup_{\gamma \in F} W_\gamma$$

has the property that if an object $E \in \mathcal{D}$ is semistable for some stability condition in C, then it is semistable for all stability conditions in C.

The following theorem is due to Toda [Tod13] and shows that the stability conditions $\sigma_{\beta,\omega} = (Z_{\beta,\omega}, \mathcal{A}_{\beta\omega})$ satisfy the support property and so exhibit a well-behaved wall crossing.

Theorem 2.30. *[Tod13, Theorem 3.23] There exists a constant $C_\omega > 0$ depending only on the class $[\omega] \in \mathbb{P}(H^2(X)_{\mathbb{R}})$, such that if E is $\sigma_{\beta,\omega}$-semistable, then*

$$\overline{\Delta}_{\beta,\omega}(E) := (ch_1^\beta(E)\omega)^2 - 2\omega^2 ch_0(E) ch_2^\beta(E) \geq 0, \text{ and}$$
$$\Delta^C_{\beta,\omega}(E) := ch_1^\beta(E)^2\omega^2 + C_\omega (ch_1^\beta(E)\omega)^2 \geq 0,$$

where $(ch_0(E), ch_1^\beta(E), ch_2^\beta(E)) = e^{-\beta} ch(E)$. Moreover, $\ker(Z_{\beta,\omega})$ is negative definite with respect to the quadratic form $\Delta^C_{\beta,\omega}$ and therefore $\sigma_{\beta,\omega} \in \mathrm{Stab}(\mathrm{X})^*$.

As observed by Bayer, Macri, and Toda in [BMT14, Proposition 7.4.1], the first inequality in Theorem 2.30 implies a generalization of a result of Arcara and Bertram [AB13, Proposition 3.6] for surfaces with Picard number 1. The precise statement is

Proposition 2.31. *If E is a μ_ω-stable vector bundle on X with $\overline{\Delta}_{\beta,\omega}(E) = 0$, then E is $\sigma_{\beta,\omega}$-stable.*

Remark 2.32. Assume that β and ω are parallel, say $\beta = sH$ and $\omega = tH$ for some $H \in \mathrm{NS}(X)$ ample and $t > 0$. Then

$$\overline{\Delta}_{\beta,\omega} = t^2((ch_1 H)^2 - 2H^2 ch_0 ch_2).$$

Then clearly any line bundle of L with $ch_1(L) = cH$ will satisfy $\overline{\Delta}_{\beta,\omega}(L) = 0$ and therefore will be $\sigma_{sH,tH}$-stable for all pairs $(s,t) \in \mathbb{R} \times \mathbb{R}_{>0}$.

2.5 Set-theoretic wall-crossing

The results in this section seem to be known to the experts, but we decided to include here some proofs for the seek of completeness.

In [ABCH13] the authors describe what new objects become stable after crossing a wall. The idea is the following: assume that a wall for the family of stability conditions $\sigma_{D,tH}$ is produced by a destabilizing sequence

$$0 \to A \to E^+ \to B \to 0$$

and assume furthermore that A and B are stable at the wall and E^+ is stable above the wall. Then the destabilizing sequence is a Jordan-Holder filtration for the semistable object E^+ at the wall. Crossing the wall will produce semistable objects that are S-equivalent to E^+ at the wall; i.e., the new objects must have A and B as stable factors, and so they must be extensions of the form

$$0 \to B \to E^- \to A \to 0.$$

But even more is true,

Proposition 2.33. *Assume that $\mu_{D,t_0H}(A) = \mu_{D,t_0H}(B)$ for some objects $A, B \in \mathcal{A}_{DH}$ and that there is $\epsilon > 0$ such that A and B are $\mu_{D,tH}$-stable with $\mu_{D,tH}(A) < \mu_{D,tH}(B)$ for $t_0 \leq t < t_0 + \epsilon$. If $Ext^1(B, A) \neq 0$ in $\mathcal{A}_{DH}$, then there exists $\delta > 0$ such that every nontrivial extension*

$$0 \to A \to E \to B \to 0$$

is $\mu_{D,tH}$-stable, or $\mu_{D,tH}$-pseudo-stable (see Definition 3.24) when $r(E) = 0$, for all $t_0 < t < t_0 + \delta$.

Proof. Let $0 < \delta \leq \epsilon$ such that there are no walls for E between t_0 and $t_0 + \delta$ (this is possible because the walls are locally finite). It is enough to prove that there is no stable subobject $E' \hookrightarrow E$ destabilizing E. If there were such E', then at the wall $W := W_{ch(A),ch(B)}$ E' is semistable and $\mu_{D,t_0H}(E') = \mu_{D,t_0H}(E)$; otherwise it would destabilize E. The map $E' \to B$ must be surjective; otherwise it would be the zero map, and therefore we would get an inclusion $E' \hookrightarrow A$ in which case $\mu(E') < \mu(A) < \mu(E)$ above the wall. Let K be its kernel. Then there is an inclusion $K \hookrightarrow A$. Since the slopes of K and A are equal at W, then either $K = 0$ in which case the sequence $A \to E \to B$ splits or $K = A$, and therefore $E' = E$. $\square$

Moreover, the more general result holds

Proposition 2.34. *Let E be an object in $\mathcal{A}_{DH}$, which is strictly semistable for some $t_0 > 0$, and assume that E has a Jordan-Holder filtration at the wall determined by t_0 that becomes the Harder-Narasimhan filtration of E on one of the chambers determined by t_0, then E is stable (or pseudo-stable when $r(E) = 0$) on the other chamber.*

Proof. Without loss of generality we can assume that at the wall determined by t_0, E has a Jordan-Holder filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that for t sufficiently near t_0 and above the wall $F_i = E_i/E_{i-1}$ is $\mu_{D,tH}$-stable, and the sequence $\mu_{D,tH}(F_i)$ is strictly increasing. Then by applying Proposition 2.33 to the exact sequences

$$0 \to E_{i-1} \to E_i \to F_i \to 0$$

we conclude that there exist $\delta > 0$ such that each E_i is $\mu_{D,tH}$-stable for all $t \in (t_0, t_0 + \delta)$. In particular $E_n = E$ is stable for every t in this interval (or pseudo-stable if $r(E) = 0$). □

Proposition 2.35. *Let us assume that the length of a Jordan-Holder filtration for E (and so of any) is 2 at a wall determined by $t_0 > 0$ and that E fits into a diagram*

$$\begin{tikzcd} & & B' \arrow[d, hook] \\ A \arrow[r, hook] & E \arrow[r, two heads] & B \arrow[d, two heads] \\ & & B'' \end{tikzcd}$$

where A, B' and B'' are the stable factors of E. Assume that there is $\epsilon > 0$ such that for all $t \in (t_0, t_0 + \epsilon)$ A, B', B'' are $\sigma_{D,tH}$-stable and $\mu_{D,tH}(A) < \mu_{D,tH}(B') < \mu_{D,tH}(B'')$. Then there exists $\delta > 0$ such that objects $\tilde{E}$ that are extensions of the form

$$\begin{tikzcd} & & & B' & \\ 0 \arrow[r] & A \arrow[r] & \tilde{E} \arrow[r] & \tilde{B} \arrow[r] \arrow[u, two heads] & 0 \\ & & & B'' \arrow[u, hook] & \end{tikzcd}$$

can not be stable for any $t_0 < t < t_0 + \delta$.

Proof. By proposition 2.33 there exists $\delta > 0$ such that all extensions $0 \to A \to E \to B \to 0$ are $\sigma_{D,tH}$-stable for $t_0 < t < t_0 + \delta$. If $\tilde{E}$ is $\sigma_{D,tH}$-stable, then $\mu_{D,tH}(A) < \mu_{D,tH}(\tilde{E}) < \mu_{D,tH}(B')$ and so $\mathrm{Hom}(\mathrm{B}', \mathrm{A}) = 0$, which gives us an inclusion

$$\mathrm{Ext}^1(B'', A) \hookrightarrow \mathrm{Ext}^1(B, A).$$

The image of every non-zero element corresponds to a nontrivial extension, which is stable by Proposition 2.34. Such extensions admit an injective morphism $B' \hookrightarrow E$ that can be visualized in the diagram

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & B' \arrow[d] \arrow[dl, "\exists"'] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & G \arrow[r] & B'' \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}
$$

Stability of E implies $\mu_{D,tH}(B') < \mu_{D,tH}(E) = \mu_{D,tH}(\tilde{E})$, and so B' destabilizes $\tilde{E}$. Thus the only possibility is $\mathrm{Ext}^1(B'', A) = 0$, which gives a surjective map $\mathrm{Ext}^1(B', A) \twoheadrightarrow \mathrm{Ext}^1(\tilde{B}, A)$, implying that $\tilde{E}$ is a pullback of an extension of B' by A. As before, there is an injective map $B'' \hookrightarrow \tilde{E}$

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & B'' \arrow[d] \arrow[dl, "\exists"'] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & \tilde{E} \arrow[r] \arrow[d] & \tilde{B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & \tilde{G} \arrow[r] & B' \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}
$$

Again the stability of E implies

$$\mu_{D,tH}(B'') > \mu_{D,tH}(E) = \mu_{D,tH}(\tilde{E})$$

destabilizing $\tilde{E}$. $\square$

For the following Corollary assume that there is a small open interval $J \subset (0, +\infty)$ such that the family stability conditions $\{\sigma_{D,tH}\}_{t \in J}$ have coarse moduli spaces.

Corollary 2.36 (Set-theoretic wall-crossing). *Let $0 \to A \to E \to B \to 0$ be an exact sequence in $\mathcal{A}_{DH}$ producing a wall $W := W_{ch(A),ch(E)}$ at $t_0 \in J$. Then*

(a) There exists $\delta > 0$ such that A and B and so E are $\sigma_{D,tH}$-stable (or pseudo-stable when $r(E) = 0$) for all $t_0 < t < t_0 + \delta$, and the Bridgeland moduli spaces for the invariants $ch(A)$ and $ch(B)$ are constant for all such t.

(b) Denote by $M_{D,t^+H}(ch(A))$ and $M_{D,t^+H}(ch(B))$ the Bridgeland moduli spaces above and sufficiently near W. Crossing W interchanges extensions

$$0 \to A^+ \to E \to B^+ \to 0 \quad A^+ \in M_{D,t^+H}(ch(A)), B^+ \in M_{D,t^+H}(ch(B))$$

by extensions

$$0 \to B^- \to F \to A^- \to 0 \quad B^- \in M_{D,t^-H}(ch(B)), A^- \in M_{D,t^-H}(ch(B))$$

where $M_{D,t^-H}(ch(B))$ and $M_{D,t^-H}(ch(A))$ denote the Bridgeland moduli spaces below and sufficiently near W.

Proof. By induction on the number of stable factors at the wall and by Proposition 2.35, we have that semistable objects at the wall satisfy the hypothesis of Proposition 2.34. □

CHAPTER 3

STABILITY CONDITIONS ON $\mathbb{P}^2$

We now concentrate in the case $X = \mathbb{P}^2$. In this case (Picard number 1), one can treat $ch(E)$ as a vector with numerical entries. Choosing $\omega = H$, the hyperplane class, and $\beta = sH$, the central charge takes the form

$$Z_{s,t}(ch_0, ch_1, ch_2) = -d_{s,t} + \sqrt{-1}r_{s,t}$$

where

$$\begin{aligned} d_{s,t} &= ch_2 - ch_1 s + \frac{ch_0}{2}(s^2 - t^2) \text{ and} \\ r_{s,t} &= t(ch_1 - ch_0 s). \end{aligned}$$

Since the category $\mathcal{A}_{\beta\omega}$ depends only on s, we denote it by $\mathcal{A}_s$ and the corresponding stability condition by $\sigma_{s,t}$. One of the most important results in [ABCH13] is the following:

Theorem 3.1 ([ABCH13])**.** *There are projective coarse moduli spaces $M_{s,t}(v)$ classifying S-equivalence classes of families of $\sigma_{s,t}$-semistable objects in $\mathcal{A}_s$ of topological type v.*

The idea is to identify $\sigma_{s,t}$-stability with quiver stability. Let $k \in \mathbb{Z}$, and consider the extension closure

$$\mathcal{A}(k) = \langle \mathcal{O}(k-2)[2], \mathcal{O}(k-1)[1], \mathcal{O}(k) \rangle.$$

An element of $\mathcal{A}(k)$ is a complex

$$\mathbb{C}^{n_0} \otimes \mathcal{O}(k-2) \to \mathbb{C}^{n_1} \otimes \mathcal{O}(k-1) \to \mathbb{C}^{n_2} \otimes \mathcal{O}(k).$$

The vector $\mathfrak{n} = (n_0, n_1, n_2)$ is its dimension vector. Let $\mathfrak{a}$ be a vector orthogonal to $\mathfrak{n}$. An object of dimension vector $\mathfrak{n}$ is said to be quiver (semi)stable with respect to $\mathfrak{a}$ if for any subcomplex in $\mathcal{A}(k)$ of dimension vector $\mathfrak{n}'$, one has $\mathfrak{n}' \cdot \mathfrak{m}(\geq) > 0$.

Moduli spaces of quiver semistable complexes of fixed dimension vector with respect to a fixed polarization have a construction via GIT given in [Kin94]. Proposition 7.3 of [ABCH13] shows that for every (s,t) in the region

$$(s-(k-1))^2+t^2<1$$

there exists a choice of a polarization $\mathfrak{a}_{s,t}$ such that moduli spaces of $\sigma_{s,t}$-stable objects are isomorphic to moduli spaces of stable objects in $\mathcal{A}(k)$ with respect to $\mathfrak{a}_{s,t}$, and for each $\sigma_{s,t}$-semistable object E of topological type v, either E or $E[1]$ lies in $\mathcal{A}(k)$. The potential walls foliate the (s,t) plane, and every potential wall intersects one of the quiver regions above. Then since the moduli of stable objects remains unchanged along every potential wall one has that for every (s,t) and every choice of invariants the moduli spaces $M_{s,t}(v)$ are projective and may be constructed by GIT.

The change from Chern classes to dimension vectors is given by the matrix

$$\begin{bmatrix} \frac{k(k-1)}{2} & \frac{-(2k-1)}{2} & 1 \\ k(k-2) & -(2k-2) & 2 \\ \frac{(k-1)(k-2)}{2} & \frac{-(2k-3)}{2} & 1 \end{bmatrix}$$

On the quiver moduli space, there is a natural ample line bundle defined as follows. A family of complexes on $\mathbb{P}^2$ parametrized by a scheme S is a complex:

$$U(k-2)\longrightarrow V(k-1)\longrightarrow W(k)$$

on $S\times\mathbb{P}^2$, where U,V,W are vector bundles of ranks n_0,n_1,n_2 pulled back from S, twisted, respectively, by the pullbacks of $\mathcal{O}_{\mathbb{P}^2}(k-2),\mathcal{O}_{\mathbb{P}^2}(k-1),\mathcal{O}_{\mathbb{P}^2}(k)$.

In this setting, the determinant line bundle on S

$$(\wedge^{n_0}U)^{a_0}\otimes(\wedge^{n_1}V)^{a_1}\otimes(\wedge^{n_2}W)^{a_2}$$

is the pull back of the ample line bundle on the moduli stack of complexes that restricts to the ample line bundle on the moduli space of semistable complexes determined by Geometric Invariant Theory.

Remark 3.2. In a preprint by Li and Zhao [LZ13], the authors show that the Bridgeland moduli spaces for Chern character vector $v=(1,0,-n)$ on $\mathbb{P}^2$ are irreducible and smooth. Their proof takes full advantage of the quiver construction of the moduli spaces. We remark here that their statements remain true for Bridgeland moduli spaces of rank-zero objects on $\mathbb{P}^2$ of class v with $g.c.d.(c_1(v), \chi(v)) = 1$. Even without the assumption of having a primitive class, it is still true that the moduli spaces are smooth at points representing stable objects. The proofs are similar, and we omit them here.

3.1 Wall and chamber structure

Recall from the previous chapter that a stability condition $\sigma = (Z, \mathcal{A})$ is on a wall if there is an object $E \in \mathcal{A}$ that is semistable for σ and it is stable for stability conditions in some chamber and unstable for stability conditions on another.

In the case of $\mathbb{P}^2$ and the stability conditions $\sigma_{s,t}$, a wall for a Chern character v is produced when there is an object E with $ch(E) = v$ and an inclusion $A \hookrightarrow E$ in some $\mathcal{A}_{s_0}$ such that

$$\mu_{s_0,t}(A) = \mu_{s_0,t}(E).$$

Using the explicit formula for $\mu_{s,t}$ it is proven in [ABCH13] that the walls are nested semicircles in the (s,t)-upper half plane with center on the real axis. Denote by $W_{ch(A),ch(E)}$ the wall corresponding to the inclusion $A \hookrightarrow E$.

Lemma 3.3. *[ABCH13, Lemma 6.3] Let E be a coherent sheaf on $\mathbb{P}^2$, which is either a torsion sheaf supported in codimension 1 or a torsion-free sheaf (not necessarily Mumford-semistable) satisfying the Bogomolov inequality*

$$ch_2(E) < \frac{ch_1(E)^2}{2r(E)}$$

and suppose $A \to E$ is a map of coherent sheaves which is an inclusion of σ_{s_0,t_0}-semistable objects of $\mathcal{A}_{s_0}$ of the same slope for some

$$(s_0, t_0) \in W := W_{ch(A),ch(E)}.$$

Then $A \to E$ is an inclusion of $\sigma_{s,t}$-semistable objects of $\mathcal{A}_s$ of the same slope for every point $(s,t) \in W$.

Remark 3.4. The proof of Lemma 3.3 relies on the fact that on $\mathbb{P}^2$ the walls are nested semicircles. By results of Maciocia [Mac12] this applies to any smooth complex surface for a variety of two dimensional families of stability conditions. Lemma 3.3 has been used in this more general setting by Arcara and Miles [AM14] when studying Bridgeland stability of line bundles. This Lemma is of remarkable importance because it implies that a Jordan-Holder filtration of a strictly semistable object at a wall remains constant along the wall. It is known as Bertram's nested wall theorem. □

Remark 3.5. Lemma 3.3 was used in [ABCH13] to provided specific bounds on the radius of the walls and via an identification of $\sigma_{s,t}$-stability with quiver stability. It is shown that if E is a Mumford stable torsion-free sheaf of primitive Chern vector $ch(E)$, then there are finitely many isomorphism types of moduli spaces of $\sigma_{s,t}$-stable objects with invariants $ch(E)$, i.e., finitely many walls intersecting the (s,t)-slide. In Chapter 4 we will see that similar results are obtained for 1-dimensional sheaves. □

The following is the main result of this chapter. We postpone its proof to the last section.

Theorem 3.6. *[BMW14, Theorem 1.1] Let $X = \mathbb{P}^2$ polarized by the hyperplane class H and $M_H(v)$ be the coarse moduli space parametrizing S-equivalence classes of Gieseker semistable sheaves of primitive topological type $v = (r, c_1, ch_2)$.*

(a) When $r \neq 0$ and $M_H(v)$ is nonempty, then it is a smooth weak Fano variety (i.e., $-K_{M_H(v)}$ is big and nef) of Picard rank at most 2. In particular, it is a Mori dream space.

(b) When $r = 0$ and $c_1 > 0$, then $M_H(v)$ is nonempty of Picard rank at most 2 and a Mori dream space.

(c) Starting from $t \gg 0$ and decreasing t corresponds to running a directed MMP on $M_H(v)$. As long as the generic point of the exceptional loci of each contraction is a sheaf, then each birational model (both in the interior of the Mori chamber and at the wall) that we get in the directed MMP is isomorphic to the normalization of the

main component of the Bridgeland moduli in the corresponding Bridgeland chamber and wall.

3.2 The Gieseker moduli on Del Pezzo Surfaces

Now let X be a smooth Del Pezzo surface and use the anticanonical bundle $H = -K_X$ as the polarization. For the rest of this chapter, Gieseker or Mumford stability are all with respect to this polarization. Fix a primitive topological type $v = (r, c_1, ch_2) \in H^*(X, \mathbb{Q})_{alg}$. Consider the coarse moduli space $M(v)$ parametrizing S-equivalent classes of Gieseker semistable torsion free sheaves on X with topological type v. Suppose $M(v)$ is nonempty and irreducible (e.g., $v = (1, 0, -n)$ is the case of Hilbert scheme of n-point on X, when $X = \mathbb{P}^2$, $M(v)$ is always irreducible, c.f. [LP97] chapter 17). We will show in this section that $M(v)$ is smooth and weak Fano (i.e., $-K_{M(v)}$ is big and nef). In particular, by [BCHM10], $M(v)$ is a Mori dream space, and there is a finite rational polyhedra decomposition of the pseudo-effective cone $\overline{NE}^1(M(v)) \subset N^1(M(v))_{\mathbb{R}}$ according to the stable base locus of the divisors. The results in this section are well known to the experts and are implicitly comtained in the standard references [HL10] and [LP97].

Remark 3.7. The condition that v being primitive means $g.c.d.(r, c_1 \cdot H, \chi) = 1$, where $\chi = r\chi(\mathcal{O}_X) - c_1 \cdot K_X/2 + ch_2$ is the Euler characteristic of the sheaf. This condition implies that there are no strictly semistable sheaves; in other words, $M(v)^s = M(v)$. v being primitive also implies that there exists a universal sheaf $\mathcal{E}$ on $M(v)$ (cf., [HL10, Corollary 4.6.7]).

Let F be a stable sheaf on X with topological type v. From deformation theory we know that $\mathrm{Ext}^1_{\mathcal{O}_X}(F, F)$ is the tangent space for the deformation functor of F and $\mathrm{Ext}^2_{\mathcal{O}_X}(F, F)$ is an obstruction space. Since F is stable and X is Del Pezzo,

$$\mathrm{Ext}^2_{\mathcal{O}_X}(F, F)^\vee \cong \mathrm{Hom}_{\mathcal{O}_X}(F, F \otimes_{\mathcal{O}_X} K_X) = 0. \tag{3.1}$$

This means that $M^s(v) = M(v)$ is smooth of dimension

$$\mathrm{ext}^1_{\mathcal{O}_X}(F, F) = 1 - r^2 - 2rch_2 + c_1^2$$

by Riemann-Roch.

Remark 3.8. If F is strictly semistable, we can not conclude $M(v)$ is smooth at $[F]$ even if $\mathrm{Ext}^2_{\mathcal{O}_X}(F,F) = 0$. The issue here is that $M(v)$ is just a coarse moduli space.

3.2.1 Determinant line bundles on $M(v)$

In this subsection, we briefly review the determinant line bundle construction on $M(v)$. We will describe a general method for associating to a flat family of coherent sheaves a determinant line bundle on the base of the family. We also describe a particular determinant line bundle $\mathcal{L}_1$ on $M(v)$ such that the linear series $|\mathcal{L}_1^m|$ for large m contracts certain parts of the moduli space and defines a morphism from $M(v)$ to the Donaldson-Uhlenbeck compactification $M^{\mu ss}(v)$ of the moduli space of μ-stable vector bundles. This subsection is based on the work of J. Le Potier and J. Li. We refer to [HL10] Chapter 8 for an excellent exposition on this subject.

The Grothendieck group $K(X)$ of coherent sheaves on X becomes a ring with $1 = [\mathcal{O}_X]$ and multiplication $[F_1] \cdot [F_2] = [F_1 \otimes^L F_2]$. There is also a natural pairing χ on $K(X)$ defined to be $\chi([F_1], [F_2]) = \chi([F_1] \cdot [F_2]) = \int_X ch(F_1)ch(F_2)td(X)$. Since X is a Del Pezzo surface, the Chern character map $ch : K(X)_{\mathbb{Q}} \to H^*(X, \mathbb{Q})_{alg}$ is an isomorphism, and χ is a nondegenerate pairing on $K(X)_{\mathbb{Q}}$. Due to this isomorphism, we will occasionally abuse the notation by thinking of χ as a nondegenerate pairing on $H^*(X)_{alg}$ or sometimes even writing $\chi(u, v)$ for $u \in K(X)$, $v \in H^*(X)_{alg}$.

Let $\mathcal{E}$ be a flat family of sheaves of topological type v on X parametrized by S. Denote $[\mathcal{E}]$ its class in $K^0(X \times S)$. Denote the projection from $X \times S$ to X and S as in the diagram below.

$$\begin{tikzcd} X \times S \arrow[r, "q"] \arrow[d, "p"] & X \\ S & \end{tikzcd}$$

Notice that p is a smooth morphism, so $p_! : K^0(X \times S) \to K^0(S)$ is well defined.

Definition 3.9. Define $\lambda_{\mathcal{E}} : K(X) \longrightarrow \mathrm{Pic}(S)$ be the composition of the homomorphisms:

$$K(X) \xrightarrow{q^*} K^0(X \times S) \xrightarrow{\cdot[\mathcal{E}]} K^0(X \times S) \xrightarrow{p_!} K^0(S) \xrightarrow{det} Pic(S).$$

Notice that $\lambda_{\mathcal{E}}$ is just the 'Fourier-Mukai transform' with kernel $\mathcal{E}$ on the K-group level, composed with the determinant homomorphism, which associates to a finite complex of locally free sheaves $F^\bullet$ on S its determinant line bundle $\otimes_i (detF^i)^{(-1)^i}$.

If L is a line bundle on S, it is easy to check that

$$\lambda_{\mathcal{E}\otimes p^*L}(u) \cong \lambda_{\mathcal{E}}(u) \otimes L^{\chi(u,v)}. \tag{3.2}$$

We now apply this construction to the universal sheaf $\mathcal{E}$ on $X \times M(v)$. The universal sheaf is only well defined up to tensoring with the pull back of a line bundle from the base. If we choose $u \in v^\perp \subset K(X)$ with respect to χ, by (3.2), $\lambda_{\mathcal{E}}(u)$ will not depend on the ambiguity of the choice of the universal sheaf and therefore yields a line bundle on $M(v)$. We will simply write $\lambda(u)$ for this determinant line bundle on $M(v)$.

Remark 3.10. There is no universal sheaf on the coarse moduli space $M(v)$ in general. The determinant line bundle is just a line bundle on the moduli stack. This line bundle, however, always descends to $M^s(v)$ for $u \in v^\perp$. If $M^s(v) \subsetneq M(v)$, one needs to put extra conditions on u to guarantee that this line bundle descends to the whole coarse moduli space $M(v)$ (cf., [HL10, Theorem 8.1.5]). If $X = \mathbb{P}^2$, every line bundle on $M^s(v)$ can be extended to $M(v)$ since $M(v)$ is locally factorial. For torsion free sheaves, this is Theorem 3.16, and for torsion sheaves of dimension 1, it is proved in [LP93].

There are two distinguished determinant line bundles on $M(v)$ given by taking

$$u_i = -r \cdot h^i + \chi(v, h^i)[\mathbb{C}_x] \in v^\perp,\ i = 0, 1 \tag{3.3}$$

where $h = [\mathcal{O}_H] \in K(X)$, $x \in X$, and $\mathcal{L}_i := \lambda(u_i)$ does not depend on the choice of the point x.

It is proved (cf., [Li93] and [HL10, 8.2]) that for large m, the linear systerm $|\mathcal{L}_1^m|$ is base point free and gives a morphism from $M(v)$ to the Donaldson-Uhlenbeck compactification $M^{\mu ss}(v)$ of the moduli space of μ- semistable sheaves. Two stable torsion free sheaves F_1, F_2 define the same point in $M^{\mu ss}(v)$ if and only if $F_1^{**} \cong F_2^{**}$ and F_1^{**}/F_1 has the same length as F_2^{**}/F_2 and gives the same point in the suitable symmetric product of X. As a consequence, $\mathcal{L}_1$ is big and nef (but not ample).

3.2.2 The canonical class of $M(v)$

In this subsection, we will prove that the anticanonical bundle $-K_{M(v)}$ is numerically equivalent to $\mathcal{L}_1$ and therefore is big and nef. Let $\mathcal{E}$ be a universal sheaf on $M(v)$ and p be the projection from $X \times M(v)$ to $M(v)$. Denote

$$\mathcal{H}om_p(\mathcal{E}, -) = p_* \circ \mathcal{H}om(\mathcal{E}, -) \colon \mathrm{Coh(X \times M(v))} \longrightarrow \mathrm{Coh(M(v))}$$

the relative Hom functor and

$$\mathcal{E}xt_p^{\bullet}(\mathcal{E}, -) = Rp_* \circ R\mathcal{H}om(\mathcal{E}, -)$$

its derived functor. The Kodaira-Spencer map naturally indentifies tangent bundle $T_{M(v)}$ with $\mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E})$, which can also be described as the sheaf associated to the presheaf

$$U \longrightarrow \mathrm{Ext}^1(\mathcal{E}|_{\mathrm{p^{-1}(U)}}, \mathcal{E}|_{\mathrm{p^{-1}(U)}}).$$

It suffices to prove that $c_1(\mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E})) = c_1(\mathcal{L}_1)$. This is an application of the Grothendieck-Riemann-Roch theorem.

Lemma 3.11. *$\mathcal{H}om_p(\mathcal{E}, \mathcal{E}) \cong \mathcal{O}_{M(v)}$, $\mathcal{E}xt^2_p(\mathcal{E}, \mathcal{E}) = 0$.*

Proof. Since the restriction of $\mathcal{E}$ to any fiber of p is stable, and there is a nowhere vanishing section of $\mathcal{H}om_p(\mathcal{E}, \mathcal{E})$, namely the identity map on the fibers of p, the first statement follows. The second statement follows from (3.1). □

In the Grothendieck group $K(M(v))$,

$$[\mathcal{E}xt^{\bullet}_p(\mathcal{E}, \mathcal{E})] = [\mathcal{H}om_p(\mathcal{E}, \mathcal{E})] - [\mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E})] + [\mathcal{E}xt^2_p(\mathcal{E}, \mathcal{E})].$$

By lemma 3.11,

$$c_1(\mathcal{E}xt^1_p(\mathcal{E}, \mathcal{E})) = -c_1(\mathcal{E}xt^{\bullet}_p(\mathcal{E}, \mathcal{E})) = -c_1(Rp_*(\mathcal{E}^{\vee} \otimes^L \mathcal{E}))$$

where $\mathcal{E}^{\vee}$ stands for the derived dual $R\mathcal{H}om^{\bullet}(\mathcal{E}, \mathcal{O}_{X \times M(v)})$.

Applying Grothendieck-Riemann-Roch to the product family $X \times M(v)$, we have equalities in $H^*(X, \mathbb{Q})_{alg}$,

$$
\begin{aligned}
&c_1(Rp_*(\mathcal{E}^\vee \otimes^L \mathcal{E})) \\
&= p_*\{(ch(\mathcal{E}^\vee \otimes^L \mathcal{E}) \cdot q^* td(X))\}_3 \\
&= p_*\{(ch(\mathcal{E})^\vee \cdot ch(\mathcal{E}) \cdot q^* td(X))\}_3 \\
&= p_*\{(r, -c_1(\mathcal{E}), ch_2(\mathcal{E}), -ch_3(\mathcal{E})) \cdot (r, c_1(\mathcal{E}), ch_2(\mathcal{E}), ch_3(\mathcal{E})) \cdot q^*(1, \frac{-K_X}{2}, \chi_X)\}_3 \\
&= p_*\{(r^2, 0, 2rch_2(\mathcal{E}) - c_1^2(\mathcal{E}), 0) \cdot q^*(1, \frac{-K_X}{2}, \chi_X)\}_3 \\
&= p_*\{(2rch_2(\mathcal{E}) - c_1^2(\mathcal{E})) \cdot \frac{-q^*K_X}{2}\}
\end{aligned}
$$

Here $\{\}_3$ means taking complex degree 3 part of a cohomology class, and p_* is the Gysin map.

On the other hand,

$$
\begin{aligned}
ch(u_1) &= -r(0, H, -\frac{H^2}{2}) + (c_1 \cdot H)(0,0,1) \\
&= (0, -rH, \frac{rH^2}{2} + c_1 \cdot H), \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
c_1(\lambda(u_1)) &= c_1(p_!(q^*(u_1) \cdot [\mathcal{E}])) \\
&= p_*\{q^* ch(u_1) \cdot ch(\mathcal{E}) \cdot q^* td(X)\}_3 \\
&= p_*\{q^*(0, -rH, \frac{rH^2}{2} + c_1 \cdot H) \cdot q^*(1, \frac{H}{2}, \chi(\mathcal{O}_X)) \cdot ch(\mathcal{E})\}_3 \\
&= p_*\{q^*(0, -rH, c_1 \cdot H) \cdot ch(\mathcal{E})\}_3 \\
&= p_*\{-rq^*H \cdot ch_2(\mathcal{E}) + c_1(\mathcal{E}) \cdot q^*(c_1 \cdot H)\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&c_1(\lambda(u_1)) - c_1(\mathcal{E}xt_p^1(\mathcal{E}, \mathcal{E})) \\
&= c_1(\lambda(u_1)) + c_1(\mathcal{E}xt_p^\bullet(\mathcal{E}, \mathcal{E})) \\
&= p_*\{-\frac{1}{2}c_1^2(\mathcal{E}) \cdot q^*H + c_1(\mathcal{E}) \cdot q^*(c_1 \cdot H)\}.
\end{aligned} \tag{3.4}
$$

If we write $c_1(\mathcal{E}) = p^*(c_1(Q)) + q^*(c_1)$ for some line bundle Q on $M(v)$, then (3.4) becomes

$$
\begin{aligned}
&p_*\{-\frac{1}{2}(p^*c_1^2(Q) + q^*c_1^2 + 2p^*c_1(Q)q^*c_1) \cdot q^*H + (p^*c_1(Q) + q^*c_1) \cdot q^*(c_1 \cdot H)\} \\
&= p_*\{-\frac{1}{2}p^*c_1^2(Q) \cdot q^*H\},
\end{aligned}
$$

and clearly

$$p_*(-\frac{1}{2}p^*c_1^2(Q)\cdot q^*H)=0,$$

as desired.

As explained at the end of Section 3.2.1, $\mathcal{L}_1$ is big and nef. Combining this with the computation above, we obtain

Proposition 3.12. *Suppose the Gieseker moduli space $M_H(v)$ of primitive topological type v on a smooth Del Pezzo surface X is nonempty and irreducible, then $M_H(v)$ is a smooth weak Fano variety. In particular, it is a Mori dream space.*

3.2.3 The Gieseker moduli on $\mathbb{P}^2$

A lot more is known in the case $X=\mathbb{P}^2$. First we have an explicit description on v such that $M_H(v)$ is nonempty. Secondly, $M_H(v)$ is always irreducible and locally factorial, and we have a explicit description of its Picard group thanks to the work of Drézet and Le Potier [Dre88], [DLP85]. These properties allow us to remove the assumption that v is of primitive type. In this subsection, we summarize these properties of $M_H(v)$ (for arbitrary v), which will be used in this paper and refer to [LP97] for proofs.

Definition 3.13. A (semi)stable sheaf F on $\mathbb{P}^2$ is called (semi)exceptional if

$$\mathrm{Ext}^1(\mathrm{F},\mathrm{F})=0.$$

All exceptional sheaves are bundles, and if there exists an exceptional bundle of topological type v, then $M_H(v)$ is reduced to a point.

Drézet and Le Potier [DLP85] gave a necessary and sufficient condition for the existence of semistable torsion free sheaves of fixed topological type. Using their result, Drézet [Dré87] constructed a function $\delta:\to[\frac{1}{2},1]$, which is periodic of period 1 and Lipschitz-continuous (cf., [LP97] Chapter 16 for the precise definition of δ), such that

Theorem 3.14. ([*Dré87*]) *The necessary and sufficient condition for the existence of nonexceptional semistable sheaf of slope μ and discriminant $\Delta=c_1^2-2rch_2$ is*

$$\Delta\geq\delta(\mu).$$

Remark 3.15. An analog of the above Theorem for torsion sheaves of dimension 1 is proved in [LP93].

If $M_H(v)$ is nonempty, we also know the following properties,

Theorem 3.16. *([Dre88], [DLP85]) If nonempty, $M_H(v)$ is always irreducible, normal and locally factorial.*

If $\Delta > \delta(\mu)$, the complement of $M^s_H(v)$ in $M_H(v)$ is of codimension at least 2, and the homomorphism in definition 3.9

$$\lambda : v^{\perp} \to Pic(M^s_H(v)) \cong Pic(M_H(v))$$

is an isomorphism.

If $\Delta = \delta(\mu)$, $Pic(M_H(v))$ is free abelian group of rank 1.

Remark 3.17. If $\Delta = \delta(\mu)$, it could happen that the complement of $M^s_H(v)$ in $M_H(v)$ is of codimension 1. Nevertheless, the homomorphism λ is still well defined and is an epimorphism (cf., [LP97] 18.3).

If $\Delta = \delta(\mu)$, $M_H(v)$ is automatically a Mori dream space since its Picard group is free abelian of rank 1. We will be mostly interested in the $\Delta > \delta(\mu)$ case (i.e., Picard number is 2). Although $M_H(v)$ may not be smooth, it is locally factorial and the complement of $M^s_H(v)$ has a codimension of at least 2. So the calculation of canonical class of $M^s_H(v)$ in Section 3.3 will extend to $M_H(v)$, and therefore $M_H(v)$ is still a Mori dream space.

Running a directed MMP on a Mori dream space M of Picard number 2 is straightforward. Recall that the movable cone $\overline{Mov}(M) \subset N^1(M)_{\mathbb{R}}$ is (the closure of) the cone spanned by all line bundles L whose stable base locus is of codimension at least 2 in M. We have

$$Nef(M) \subset \overline{Mov}(M) \subset \overline{NE}^1(M).$$

For any big divisor D, draw a line connecting D with an ample divisor A on M. This line will cross finitely many walls between Mori chambers. The wall crossing corresponds to two different birational models of M. There are two cases. If the wall lies in the interior of $\overline{Mov}(M)$, then the corresponding birational map $M_i \dashrightarrow$

M_j is a D-flip. If the wall happens to be one boundary of $\overline{Mov}(M)$ (but not on the boundary of $\overline{NE}^1(M)$), it corresponds to a divisorial contraction

$$M_j \to M'_j$$

on some M_j contracting some irreducible divisor B, and there is just one more Mori chamber outside of $\overline{Mov}(M)$ generated by (the pullback to M of) $Nef(M'_j)$ and (the strict transform of) B. Thus if $D \in \overline{Mov}(M)$, after finitely many D-flips, we ended up with a model birational model M_j on which (the strict transform of) D is semiample, or if $D \notin \overline{Mov}(M)$, after finitely many D-flips, we have a divisorial contraction

$$\pi: \ M_j \to M'_j$$

such that $D \in \pi^*(Nef(M'_j)) + B$.

3.3 Determinant line bundles on the Bridgeland moduli

Let $X = \mathbb{P}^2$, and assume for the moment that $v = (r, c_1, ch_2)$ is primitive. The potential wall (in the (s,t)-plane) associated to a pair of Chern characters $v = (r, c, d)$ and $v' = (r', c', d')$ is the set

$$W_{v,v'} := \{(s,t) \mid \mu_{s,t}(v) = \mu_{s,t}(v')\}.$$

We will only be interested in walls in the region where $s < \frac{c_1 \cdot H}{r}$ because a Mumford-stable torsion-free sheaf E of degree c_1 only belongs to the category $\mathcal{A}_s$ if $s < \frac{c_1 \cdot H}{r}$. A simple computation shows that the potential walls are nested semicircles in this region (see Chapter 4 for specific examples).

For each (s,t) on the potential wall $W_{v,v'}$ (there could be more than one v' giving the same wall, but $span\{v', v\}$ is determined by (s,t)), we associate a **canonical** determinant line bundle on the Bridgeland moduli in the following manner.

Let $w_{s,t}$ be an integral topological type (up to scalar) in $H^*(X, \mathbb{R})_{alg}$ perpendicular to v and v' under the nondegenerate pairing χ. Since on the plane $\Pi = span\{v', v\}$, $\mu_{s,t}$ is constant, we can choose an orientation of $w_{s,t}$ such that

$$\chi(w_{s,t}, ch(C)) > 0$$

for every object $C \in \mathcal{A}_s$ with $\mu_{s,t}(C) < \mu_{s,t}(v)$. Finally, choose a complex $F_{s,t}$ such that $ch(F_{s,t}) = w_{s,t}$.

For a flat family of $\sigma_{s,t}$-semistable complexes $\mathcal{E}$ of topological type v (or $-v$) on $\mathbb{P}^2 \times S$:

$$U(k-2) \longrightarrow V(k-1) \longrightarrow W(k) \tag{3.5}$$

either $\mathcal{E}|_b \in \mathcal{A}_s$ or $\mathcal{E}[-1]|_b \in \mathcal{A}_s$ for any $b \in S$. We associate a line bundle on S in the same way as in definition 3.9:

$$\lambda_{s,t} := \begin{cases} \det(p_!([q^*F_{s,t}] \cdot [\mathcal{E}])) & \text{if } \ \mathcal{E}|_b \in \mathcal{A}_s \\ \\ \det(p_!([q^*F_{s,t}] \cdot [\mathcal{E}[-1]])) & \text{if } \ \mathcal{E}[-1]|_b \in \mathcal{A}_s. \end{cases}$$

Lemma 3.18. *If $\mathcal{E}|_b \in \mathcal{A}_s$ for any $b \in S$, then*

$$\lambda_{s,t} = (\wedge^{n_0} U)^{a_0} \otimes (\wedge^{n_1} V)^{a_1} \otimes (\wedge^{n_2} W)^{a_2}$$

for some $\vec{a} = (a_0, a_1, a_2)$ and $a_i = (-1)^i \chi(w_{s,t}, \mathcal{O}_{\mathbb{P}^2}(k-2+i))$.

Proof. We have

$$\begin{aligned} p_!([q^*F_{s,t}] \cdot [U(k-2)]) &= p_!([q^*F_{s,t} \otimes q^*\mathcal{O}_{\mathbb{P}^2}(k-2) \otimes p^*U]) \\ &= [U \otimes p_!(q^*F_{s,t}(k-2)])], \end{aligned}$$

and

$$[p_!(q^*(F_{s,t}(k-2))] = [\mathcal{O}_S^{\oplus a_0}].$$

So the first statement is clear. To determine a_0, we apply the Grothendieck-Riemann-Roch Theorem,

$$\begin{aligned} a_0 &= ch_0(p_!(q^*(F_{s,t}(k-2))) \\ &= p_*\{q^*ch(F_{s,t}) \cdot q^*ch(\mathcal{O}_{\mathbb{P}^2}(k-2)) \cdot q^*td(T_{\mathbb{P}^2})\}_2 \\ &= p_*\{(q^*(w_{s,t} \cdot ch(\mathcal{O}_{\mathbb{P}^2}(k-2)) \cdot td(T_{\mathbb{P}^2}))\}_2 \\ &= \chi(w_{s,t}, \mathcal{O}_{\mathbb{P}^2}(k-2)). \end{aligned}$$

Similarly, one can prove the formula for the V, and W terms. $\square$

Remark 3.19. If fiber of $\mathcal{E}[-1]$ is in $\mathcal{A}_s$, we have an extra negative sign in front of the formula for a_i due to the choice of the orientation of $w_{s,t}$.

Lemma 3.18 implies that

$$\begin{aligned} a_0n_0 + a_1n_1 + a_2n_2 &= \chi(w_{s,t}, [\mathcal{O}_{\mathbb{P}^2}(k-2)^{n_0} \to \mathcal{O}_{\mathbb{P}^2}(k-1)^{n_1} \to \mathcal{O}_{\mathbb{P}^2}(k)^{n_2}]) \\ &= \chi(w_{s,t}, \pm v) = 0, \end{aligned}$$

and therefore we can talk about quiver stable objects of dimension vector $\vec{n}$ with respect to polarization $\vec{a}$.

Lemma 3.20. *If $F \in \mathcal{A}_s$, then $\chi(w_{s,t}, ch(F)) > 0$ (resp. < 0) if and only if $\mu_{s,t}(F) <$ (resp. $>$) $\mu_{s,t}(v)$.*

Proof. The plane $span\{v, v'\}$ in $H^*(X, \mathbb{Q})_{alg}$ is where the slope function $\mu_{s,t}$ equals the constant $\mu_{s,t}(v)$. The conclusion follows from the choice of orientation of $w_{s,t}$. □

Proposition 3.21. *An object $E = [\mathcal{O}_{\mathbb{P}^2}(k-2)^{n_0} \to \mathcal{O}_{\mathbb{P}^2}(k-1)^{n_1} \to \mathcal{O}_{\mathbb{P}^2}(k)^{n_2}]$ is quiver (semi)stable with respect to the polarization $\vec{a}$ if and only if it is $\sigma_{s,t}$-(semi)stable.*

Proof. Without loss of generality, let us just prove the equivalence of semistable objects under both stability conditions. Suppose E is $\sigma_{s,t}$-semistable but quiver unstable. Since the quiver heart $\mathcal{A}(k)$ is coming from tilting $\mathcal{A}_s = \mathcal{P}_{s,t}(0,1]$ with respect to the torsion pair $(\mathcal{P}_{s,t}(\alpha, 1], \mathcal{P}_{s,t}(0, \alpha])$ for some $\alpha \in (0, 1]$, $E \in \mathcal{P}_{s,t}(\alpha, 1]$ or $\mathcal{P}_{s,t}(0, \alpha][1]$.

First let us assume $E \in \mathcal{P}_{s,t}(\alpha, 1]$. Let F be a quiver destablizing subobject of E in $\mathcal{A}(k)$ with dimension vector $\vec{b}$. Then $\vec{a} \cdot \vec{b} < 0$, or equivalently, $\chi(w_{s.t}, ch(F)) < 0$. Since $\mathrm{Hom}(\mathcal{P}_{s,t}(0,\alpha][1], \mathcal{P}_{s,t}(\alpha, 1]) = 0$, $F \in \mathcal{P}_{s,t}(\alpha, 1]$ as well. Let $F' \in \mathcal{P}_{s,t}(\alpha, 1] \subset \mathcal{A}(k)$ be the first $\sigma_{s,t}$-semistable factor of F, then by Lemma 3.20, $\mu_{s,t}(F') \geq \mu_{s,t}(F) > \mu_{s,t}(E)$. Since E is $\sigma_{s,t}$-semistable, the composition

$$F' \to F \to E$$

is zero. This contradicts the fact that F is a subobject of E in $\mathcal{A}(k)$. If $E \in \mathcal{P}_{s,t}(0, \alpha][1]$, we could consider a quiver destabilizing quotient H of E in $\mathcal{A}(k)$. A similar argument as above gives $H \in \mathcal{P}_{s,t}(0,\alpha][1]$ as well. The rest of the argument can be done similarly to the previous case by considering the last $\sigma_{s,t}$-semistable factor H' of H.

Conversely, suppose E is quiver semistable. Notice that the $\sigma_{s,t}$-phase $\phi_{s,t}(E)$ of E is a well defined number in $(\alpha, \alpha+1]$ (although we do not know if E is $\sigma_{s,t}$-semistable, we can still talk about the phase of $Z_{s,t}(E)$). Write E uniquely as an extension

$$P[1] \to E \to Q$$

where $P \in \mathcal{P}_{s,t}(0,\alpha]$, and $Q \in \mathcal{P}_{s,t}(\alpha,1])$. We claim that if $\alpha < \phi_{s,t}(E) \leq 1$, then $P = 0$, whereas if $1 < \phi_{s,t}(E) \leq 1+\alpha$, $Q = 0$. For the first case, let $\vec{b}$ be the dimension vector of $P[1]$. Since E is quiver semistable, $\vec{a} \cdot \vec{b} \geq 0$, and therefore by Lemma 3.18, $\chi(w_{s,t}, ch(P[1])) \geq 0$ or equivalently $\chi(w_{s,t}, ch(P)) \leq 0$, by Lemma 3.20, this implies that $\phi_{s,t}(P) \geq \phi_{s,t}(E)$, a contradiction unless $P = 0$. The second case can be treated similarly. Notice that in the second case $ch(E) = -v$. If $Q \neq 0$, let $\vec{c}$ be the dimension vector of Q. Then $\vec{a} \cdot \vec{c} \leq 0$, or equivalently, $\chi(w_{s,t}, ch(Q)) \geq 0$ because in this case there is an extra negative sign in the formula in Lemma 3.18. This implies that $\phi_{s,t}(Q) \leq \phi_{s,t}(E) - 1$, contradiction.

So again either $E \in \mathcal{P}_{s,t}(\alpha,1]$ or $E \in \mathcal{P}_{s,t}(0,\alpha][1]$. Suppose $E \in \mathcal{P}_{s,t}(\alpha,1]$, the other case can be treated similarly. Let $E' \in \mathcal{P}_{s,t}(\alpha,1]$ be the first $\sigma_{s,t}$-semistable factor of E with dimension vector $\vec{b}'$. Form the exact triangle

$$E' \to E \to H, \tag{3.6}$$

then $H \in \mathcal{P}_{s,t}(\alpha,1]$. Thus (3.6) is an exact sequence in $\mathcal{A}(k)$. Since E is quiver semistable, $\vec{a} \cdot \vec{b}' \geq 0$, again by Lemma 3.20, $\mu_{s,t}(E') \leq \mu_{s,t}(E)$, so $E' = E$ is semistable.

□

3.4 The Bridgeland moduli as birational models of $M_H(v)$

The determinant line bundle $\lambda_{s,t}$ is always ample on the Bridgeland moduli space $M_{s,t}(v)$. Notice that if there exists $U \subset M_H(v)$, an open subset whose complement is of codimention at least 2 and $\mathcal{E}$ be a flat family of sheaves on $X \times U$, which is both (s,t)-stable and Gieseker stable, then by construction of determinant line bundles,

$$\lambda_{s,t} \cong \lambda_{\mathcal{E}}(w_{s,t})$$

as line bundles on U.

This means the ample determinant line bundle $\lambda_{s,t}$ on $M_{s,t}(v)$ pulls back to $\lambda(w_{s,t})$ on $M_H(v)$. Denote $M^P_{s,t}$ the normalization of the main component of $M_{s,t}(v)$ (with reduced induced scheme structure) whose generic point corresponds to a sheaf and still denote $\lambda_{s,t}$ as the restriction to it of the ample determinant line bundle. Then we can interpret $M^P_{s,t}(v)$ as birational models of $M_H(v)$.

We are now ready to prove our main Theorem.

Proof of **Theorem 3.6**. Part (a) is just the content of Proposition 3.12. We only need to prove part (b).

To this end, consider the Bridgeland wall and chamber structure with respect to the main component in the (s,t)-plane for $t > 0$, and $s < \frac{c_1 \cdot H}{r}$. Since the walls are nested semicircles, we could choose an s such that the ray

$$\Lambda_s := \{(s,t) | t > 0\}$$

intersects all **actual** walls. Then we can decrease the parameter t, then $\lambda(w_{s,t})$ moves in $N^1(M_H(v))_{\mathbb{R}}$. An easy computation shows that $\lambda(w_{s,t})$ is moving toward the side of nef cone opposite to $-K_{M_H(v)}$. This corresponds to running a directed MMP on $M_H(v)$, and we get $M^P_{s,t}(v)$ as the birational models of $M_H(v)$.

We know that for $t \gg 0$, a sheaf F is Gieseker stable if and only if it is $\sigma_{s,t}$-stable. Thus

$$M^P_{s,t}(v) \cong M_H(v) \text{ for } t \gg 0,$$

and $\lambda_{s,t} \cong \lambda(w_{s,t})$ is ample on $M_H(v)$.

The first time (s,t) hits an actual wall at (s,t_0), by the definition of wall, every (s,t_0^+) semistable objects is still (s,t_0) semistable, but some (s,t_0^+)-stable objects become strictly (s,t_0)-semistable. By the universal property of coarse moduli space, taking S-equivalence classes of (s,t_0)-semistable objects corresponds to a contracting morphism. It follows from Lemma 3.22 that the exceptional loci is positive dimensional (this is different from the case of sheaves on a K3 surface where it is possible to find fake walls; i.e., walls where the stable objects change but the moduli space itself does not, see [BM14]),

$$\pi_0^+ : M_H(v) \cong M_{s,t_0^+}(v) \longrightarrow M_{s,t_0}^P(v),$$

and λ_{s,t_0} pulls back to a nef but not ample line bundle $\lambda(w_{s,t_0})$ on $M_H(v) \cong M_{s,t_0^+}(v)$. The first actual wall corresponds to one end of the nef cone of M_{s,t_0^+} (the other end being generated by $\lambda(u_1)$).

There are several possibilities:

(a) π_0^+ is a fiber contraction. The directed MMP stops. In this case we actually have a **collapsing wall**, which means there are no semistable objects in this component whatsoever after crossing the wall.

$$M_{s,t_0^-}^P = \emptyset.$$

(b) π_0^+ is a divisorial contraction contracting Δ. Crossing the wall will not affect $M_H(v) \setminus \Delta$. If A is the destabilizing sheaf of E at (s, t_0), i.e., $\mu_{s,t_0^+}(A) < \mu_{s,t_0^+}(E)$ but $\mu_{s,t_0^-}(A) > \mu_{s,t_0^-}(E)$, and E fits into an exact sequence in $\mathcal{A}_s$,

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0. \tag{3.7}$$

As explained in Section 2.5, if both A and B are stable, then crossing the wall amounts to replace the (s, t_0^+)-stable objects E by (s, t_0^-)-stable objects E' fitting in the 'reverse' extension

$$0 \longrightarrow B \longrightarrow E' \longrightarrow A \longrightarrow 0. \tag{3.8}$$

If either A or B is strictly σ_{s,t_0}-semistable, we just have to iterate the above process for the corresponding stable factors. In any case, by Lemma 3.22,

$$\pi_0^- : M_{s,t_0^-}^P \to M_{s,t_0}^P$$

has to be a small contraction. Because M_{s,t_0}^P is Q-factorial, we must have

$$M_{s,t_0^-}^P \cong M_{s,t_0}^P.$$

(Unlike the case of sheaves on K3 surfaces, there is no bouncing wall [BM14], i.e., $M_{s,t_0^-}^P \cong M_{s,t_0^+}^P$ can not happen.) Since $M_{s,t_0^-}^P$ is of Picard number 1, the next wall has to be a collapsing wall.

(c) π_0^+ is a small contraction. Again by Lemma 3.22, π_0^- is a small contraction as well. The ample line bundle λ_{s,t_0^-} on M^P_{s,t_0^-} pulls back to $\lambda(w_{s,t_0^-})$ on $M_H(v)$. Therefore

$$M^P_{s,t_0^-} \cong \mathrm{Proj} R(M^P_{s,t_0^-}, \lambda_{s,t_0^-})) \cong \mathrm{Proj} R(M_H(v), \lambda(w_{s,t_0^-})),$$

and we get a flip

$$\begin{tikzcd} M^P_{s,t_0^+} \arrow[rr, dashed] \arrow[dr, "\pi_0^+"'] & & M^P_{s,t_0^-}. \arrow[dl, "\pi_0^-"] \\ & M^P_{s,t_0} & \end{tikzcd}$$

After crossing the wall, we can keep running the directed MMP by decreasing t further and repeating the above process. As long as a generic point E in the exceptional loci of π_0^+ is a sheaf, the destabilizing object A is a sheaf (but may not be a subsheaf in the usual sense) as well. This follows from the long exact sequence in cohomology of (3.7):

$$\mathcal{H}^{-1}(A) \hookrightarrow \mathcal{H}^{-1}(E) \to \mathcal{H}^{-1}(B) \to \mathcal{H}^0(A) \to \mathcal{H}^0(E) \twoheadrightarrow \mathcal{H}^0(B).$$

The assumption in Lemma 3.22 is still satisfied. Since there always exists a collapsing wall, as t gets small enough, the directed MMP will either end up with case (a) or (b).

□

Lemma 3.22. *Let A, B be σ_{s,t_0}-stable objects in $\mathcal{A}_s$ and $A \in \mathcal{Q}_s$ be a sheaf. Suppose that $\mu_{s.t_0}(A) = \mu_{s,t_0}(B)$ and $\mu_{s,t_0^+}(A) < \mu_{s,t_0^+}(B)$ (therefore $\mu_{s,t_0^-}(A) > \mu_{s,t_0^-}(B)$). Then* $\dim_{\mathbb{C}} Hom^1_{\mathcal{D}}(B, A) > \dim_{\mathbb{C}} Hom^1_{\mathcal{D}}(A, B)$.

Proof. By Serre duality, $\mathrm{Hom}^1_{\mathcal{D}}(A, B) \cong \mathrm{Hom}^1_{\mathcal{D}}(B, A \otimes^L \mathcal{O}(-3))^\vee$. Consider the exact sequence of sheaves on $\mathbb{P}^2$

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0$$

where C is a general smooth cubic curve. Tensoring the above sequence with B, we get an exact triangle

$$B \otimes^L \mathcal{O}_{\mathbb{P}^2}(-3) \to B \to B \otimes^L \mathcal{O}_C.$$

Applying the derived functor $\mathrm{RHom}^\bullet(A,-)$ and taking the long exact sequence in cohomology, we obtain

$$\cdots \longrightarrow \mathrm{Hom}(A, B\otimes^L \mathcal{O}_{\mathbb{P}^2}(-3)) \longrightarrow \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A, B\otimes^L \mathcal{O}_C)$$

$$\longrightarrow \mathrm{Hom}^1(A, B\otimes^L \mathcal{O}_{\mathbb{P}^2}(-3)) \longrightarrow \mathrm{Hom}^1(A,B) \longrightarrow \mathrm{Hom}^1(A, B\otimes^L \mathcal{O}_C)$$

$$\longrightarrow \mathrm{Hom}^2(A, B\otimes^L \mathcal{O}_{\mathbb{P}^2}(-3)) \longrightarrow \mathrm{Hom}^2(A,B) \longrightarrow \mathrm{Hom}^2(A, B\otimes^L \mathcal{O}_C) \longrightarrow \cdots$$

Since both A, B are σ_{s,t_0}-stable, we have $\mathrm{Hom}_{\mathcal{D}}(A,B)=0$ and

$$\mathrm{Hom}^2_{\mathcal{D}}(A, B\otimes^L \mathcal{O}_{\mathbb{P}^2}(-3))^\vee \cong \mathrm{Hom}_{\mathcal{D}}(B,A)=0.$$

Moreover, we claim that

$$\mathrm{Hom}^i_{\mathcal{D}}(A, B\otimes^L \mathcal{O}_C)=0 \text{ for } i\leq -2, \text{ and } i\geq 2.$$

Assuming the claim, we get

$$\begin{aligned}
&\hom^1(B,A)-\hom^1(A,B)\\
&=\hom^0(A,B\otimes^L\mathcal{O}_C)-\hom^1(A,B\otimes^L\mathcal{O}_C)
\end{aligned}$$

$$\begin{aligned}
&= \hom^0(A,B\otimes^L\mathcal{O}_C)-\hom^1(A,B\otimes^L\mathcal{O}_C)\\
&\geq \chi(A^\vee\otimes^L B\otimes^L\mathcal{O}_C)\\
&= \int_{\mathbb{P}^2} ch(A^\vee)\cdot ch(B)\cdot ch(\mathcal{O}_C)\cdot td(\mathbb{P}^2)\\
&= ch_0(A)ch_1(B)-ch_0(B)ch_1(A). \qquad (3.9)
\end{aligned}$$

We prove that (3.9) has to be strictly positive. First, notice that we have

$$\mu_{s,t}(A)<\mu_{s,t}(B) \text{ for } t\gg 0,$$

where

$$\mu_{s,t}(ch_0,ch_1,ch_2)=\frac{\frac{-t^2}{2}ch_0+(ch_2-sch_1+\frac{s^2}{2}ch_0)}{t(ch_1-sch_0)}$$

and for either A or B, the denominator of $\mu_{s,t}$ is strictly positive.

According to the rank of A, there are several cases:

- $ch_0(A) = 0$. Then $ch_0(B) \leq 0$ and $ch_1(A) > 0$. If $ch_0(B) = 0$, then the wall can never be crossed as we decrease t. If $ch_0(B) < 0$, we immediately get (3.9) is positive.

- $ch_0(A) > 0$. Then we have an inequality for the dominant terms
$$\frac{ch_0(A)}{ch_1(A) - sch_0(A)} \geq \frac{ch_0(B)}{ch_1(B) - sch_0(B)}.$$
The equality can not be achieved because otherwise the wall can never be crossed again. But this precisely means
$$ch_0(A)ch_1(B) - ch_0(B)ch_1(A) > 0.$$

It remains to prove the vanishing statement in the claim. Since A, B are σ_{s,t_0}-stable, we can assume that A, B both are in the quiver heart

$$\mathcal{A}(k) = \langle \mathcal{O}_{\mathbb{P}^2}(k-2)[2], \mathcal{O}_{\mathbb{P}^2}(k-1)[1], \mathcal{O}_{\mathbb{P}^2}(k) \rangle$$

for suitable k.

When $i \leq -2$ or $i \geq 3$, then for degree reasons

$$\mathrm{Hom}^i(\mathcal{O}_{\mathbb{P}^2}(k-j)[j], \mathcal{O}_C(k-j')[j']) = 0 \text{ for any } j, j' = 0, 1, 2.$$

This gives the vanishing statement for $i \leq -2$ and $i \geq 3$. When $i = 2$, since A is a sheaf, we have

$$\mathrm{Hom}^2(A, \mathcal{O}_C(k-j)[j]) = 0 \text{ for } j = 1, 2.$$

We also have

$$\mathrm{Hom}^2_{\mathcal{D}}(A, \mathcal{O}_C(k))^\vee = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_C(k), A(-3)) = 0.$$

Because $\mathcal{O}_C(k)$ is a torsion sheaf, its image has to be torsion, but if C is general, there is no nontrivial map from $\mathcal{O}_C(k)$ to any fixed torsion sheaf. □

Remark 3.23. Things are slightly different when studying the Gieseker moduli of 1-dimensional sheaves, although most of the arguments in this chapter are the same. By the work of Le Potier [LP93] we know that these moduli spaces are irreducible and locally factorial, and their Picard group is free abelian of rank at most 2 (actually 2 for $c_1(v) \geq 3$); moreover, a specific set of generators is given, namely: the determinant line bundle and the line bundle giving the support map. It is not hard then to prove that the Gieseker moduli of 1-dimensional plane sheaves of fixed invariants is also a Mori dream space. A proof can be found in [Woo13].

We notice that in general the Gieseker moduli of 1-dimensional plane sheaves is not smooth (although its singular locus has high codimension). We can fix this by making the following

Definition 3.24. An object $F \in \mathcal{A}_s$ of Chern character $ch(F) = (0, ch_1, ch_2)$ is said to be $\sigma_{s,t}$-pseudo stable if for any inclusion $A \hookrightarrow F$ in $\mathcal{A}_s$ one has

$$\mu_{s,t}(A) \leq \mu_{s,t}(F) \text{ and } \mu_{s,t}(A) = \mu_{s,t}(F) \Rightarrow ch_0(A) = 0.$$

Remark 3.25. With this new terminology it is easy to see that on any chamber one has $\sigma_{s,t}$-semistable $=$ $\sigma_{s,t}$ pseudo-stable. Also for $t \gg 0$ a sheaf is $\sigma_{s,t}$-pseudo-stable if and only if it is Gieseker semistable, and these are all the $\sigma_{s,t}$-semistable objects (see proof of Corollary 4.7). Thus, Theorem 3.6 holds for 1-dimensional plane sheaves, with no restrictions on the topological type, when replacing stable by pseudo-stable.

CHAPTER 4

WALL-CROSSING FOR 1-DIMENSIONAL PLANE SHEAVES

As mentioned in the previous chapter, moduli spaces of Gieseker semistable plane sheaves of Hilbert polynomial $P(t) = ct + \chi$ were initially studied by J. Le Potier in [LP93], where it is shown that these moduli spaces are projective, irreducible, locally factorial, and smooth at the stable points. For small values of c, it is possible to find nice stratifications of these moduli spaces by studying their resolutions; see [DM11] for $c = 4$ and [Mai11] and [Mai13] for $c = 5$ and 6. Studying a sheaf by studying its possible resolutions is same as replacing such sheaf for an equivalent element in the derived category. Indeed each strata in the stratifications given by Drézet and Maican in [DM11] and by Maican in [Mai11] and [Mai13] can be interpreted as a set of extensions in a tilted category [BMW14].

But the story goes on, as in [BMW14], each set of extensions produces a Bridgeland wall and these are all the walls for the directed MMP. The following numerical bound coming from Lemma 3.3 produces some of these sets of extensions for an arbitrary value of c even when Maican-type stratifications are unknown.

Let E be a sheaf of topological type $(0, c, d)$ with $c > 0$, and let F be a destabilizing object (which is necessarily a sheaf). Then E and F fit into an exact sequence

$$0 \to K \to F \to E.$$

By Lemma 3.3 we must have $K \in \mathcal{F}_s$ and $F \in \mathcal{Q}_s$ for all s along the wall. If $ch(F) = (r', c', d')$, then in our case where the wall is a semicircle with center $(d/c, 0)$ and radius R, this says

$$\frac{ch_1(K)}{r(K)} \leq \frac{d}{c} - R \quad \text{and} \quad \frac{c'}{r'} \geq \frac{d}{c} + R.$$

Since $r(K) = r'$ and $ch_1(K) - c' + c \geq 0$, then combining the inequalities above we get

$$R + \frac{d}{c} \leq \frac{c'}{r'} \leq \frac{d}{c} + \frac{c}{r'} - R, \tag{4.1}$$

which immediately produces

$$R \leq \frac{c}{2r'}. \tag{4.2}$$

Fix a numerical class $v = (0, d, -\frac{3}{2}d)$ with d odd. One of the key ingredients in the computation that follows is the existence of a collapsing wall. The generic element of $M_H(v)$ corresponds to a sheaf $\mathcal{F}$ satisfying $H^0(\mathcal{F}) = 0$ (see [Mai13, Proposition 6.1.1]). By using the Beillinson spectral sequence one can conclude that the general element of $M_H(v)$ has a resolution of the form

$$0 \to d\mathcal{O}(-2) \to d\mathcal{O}(-1) \to \mathcal{F} \to 0.$$

In particular $\mathcal{O}(-1) \in \mathcal{A}_{-3/2}$ produces a wall contracting an open set. The corresponding wall W_C has center $(-3/2, 0)$ and radius

$$R = \sqrt{\frac{1}{4} + \frac{2\chi}{r}} = \frac{1}{2}.$$

The complement of this open set is the theta divisor ([LP93]) and is the set of semistable sheaves that have at least one section, i.e., those that have $\mathcal{O}$ as a subobject. The corresponding wall W_Θ if a semicircle of radius $R = \frac{3}{2}$. Crossing W_Θ corresponds to a divisorial contraction, and since $M_H(v)$ has Picard number 2, then there are no walls between W_C and W_Θ. This improves our bound for the walls corresponding to flips:

Proposition 4.1. *Let A be a coherent sheaf of rank $r > 0$ and Euler characteristic χ, and let E be a coherent sheaf with $ch(E) = (0, d, -\frac{3}{2}d)$. A morphism of sheaves $A \to E$, which is an inclusion of objects in the category $\mathcal{A}_{-3/2}$, produces a wall corresponding to a flip if and only if*

$$\frac{3}{2} < \sqrt{\frac{1}{4} + \frac{2\chi}{r}} \leq \frac{d}{2r}. \tag{4.3}$$

It is useful to know whether the new objects we get after crossing a wall are stable or pseudo-stable. We can answer this in a very special case:

Proposition 4.2. *Let $v = (0, d, -3d/2)$ and assume that $E \in \mathcal{A}_{-3/2}$ is an object with $ch(E) = v$ that has a Jordan-Holder filtration of length 1 at a wall W. Then E is stable on one of the chambers determined by W.*

Proof. Assume that the Jordan-Holder filtration for E at W is

$$0 \to A \to E \to B \to 0$$

and that $\mu_{-3/2,t}(A) > \mu_{-3/2,t}(B)$ above W, then E is pseudo-stable below W by Proposition 2.33. Assume that E is not stable, then there should be a subobject $E' \hookrightarrow E$ such that $r(E') = r(E) = 0$, $c_1(E') < c_1(E) = d$ and $\chi(E') = \chi(E) = 0$. Let $K = \ker(E' \twoheadrightarrow B)$ and $c_1(E') = d - s$, then we have a diagram

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, hook] & E' \arrow[d, hook] \arrow[dr, two heads] & \\ A \arrow[r, hook] & E \arrow[r, two heads] & B \end{tikzcd}$$

If $r(B) = r', c_1(B) = c'$ and $\chi(B) = \chi'$, then

$$\begin{aligned} (r(K), c_1(K), \chi(K)) &= (-r', d - s - c', -\chi') \\ (r(A), c_1(A), \chi(A)) &= (-r', d - c', -\chi'). \end{aligned}$$

Note that in this case $d_{-3/2,t}(K) = d_{-3/2,t}(A)$ and $r_{-3/2,t}(K) = r_{-3/2,t}(A) - st$. But A is stable at W, and so it is stable for t sufficiently near W; therefore,

$$\mu_{-3/2,t}(K) < \mu_{-3/2,t}(A)$$

for t above and below W. This implies that $sd_{-3/2,t}(A) < 0$ above and below W and so $s = 0$. Thus E is stable. □

4.1 Rank-one walls

Setting $r = 1$ in (4.3), one finds the set of admissible values for the Euler characteristic of a destabilizing object producing a wall corresponding to a flip:

$$\chi = 2, \ldots, \frac{d^2 - 1}{8}.$$

The possible values for the first Chern class come from solving the inequality (4.1). The first Chern class c of a rank 1 destabilizing object with Euler characteristic $\chi = \frac{d^2-1}{8} - \ell$ must satisfy

$$\sqrt{\frac{d^2}{4} - 2\ell} - \frac{3}{2} \leq c \leq -\frac{3}{2} + d - \sqrt{\frac{d^2}{4} - 2\ell}. \tag{4.4}$$

It is easy to check that $c = \dfrac{d-3}{2}$ is always a solution. These are the invariants of a twisted ideal sheaf of a zero-subscheme Z of length ℓ. Moreover, since for generic zero-subschemes Z and W of length ℓ we have $\mathrm{Hom}(\mathcal{O}, \mathcal{I}_{\mathrm{W}} \otimes^{\mathrm{L}} \mathcal{I}_{\mathrm{Z}}(\mathrm{d})) \neq 0$, then there are nontrivial extensions

$$0 \to \mathcal{I}_Z\left((d-3)/2\right) \to E \to \mathcal{I}_W^{\vee}\left((-d-3)/2\right)[1] \to 0$$

producing sheaves $E \in \mathcal{A}_{-3/2}$, which are Bridgeland stable for $t > \sqrt{\dfrac{d^2}{4} - 2\ell}$. This proves that the number of actual rank 1 walls corresponding to flips is $\dfrac{d^2-9}{8}$.

Notice that the exceptional loci for a rank 1 flip is not irreducible in general. Indeed, the inequality (4.4) has a unique solution only when $\ell < \dfrac{d-1}{2}$. However, setting

$$\mathcal{G}_{\ell,i}^{W} := \mathcal{I}_W\left(\frac{d-3}{2} + i\right), \quad \mathrm{length}(W) = \ell + \frac{i(d+i)}{2}, \quad i \in \mathbb{Z}$$

we have

Proposition 4.3. *If $c = \frac{d-3}{2} + i$ is solution for (4.4), then so is $c = \frac{d-3}{2} - i$. Generically, the corresponding destabilizing objects are of the form $\mathcal{G}_{\ell,i}^{W}$ and $\mathcal{G}_{\ell,-i}^{Y}$, respectively. Moreover, if $E_{\ell,k}$ denotes the component containing the locus of sheaves destabilized by an object of the form $\mathcal{G}_{\ell,k}^{W}$ then $E_{\ell,-k}$ is the image of $E_{\ell,k}$ by the duality automorphism.*

Proof. This is a trivial computation of invariants. □

If we track the construction of the determinant line bundle on the Bridgeland moduli spaces following [BMW14], we obtain a decomposition of the Mori cone of $N_{\mathbb{P}^2}(d,0)$ as in Figure 4.1. Here $\mathcal{L}$ is the line bundle giving the support map $N_{\mathbb{P}^2}(d,0) \to |\mathcal{O}_{\mathbb{P}^2}(d)|$, and $\mathcal{D}$ is the determinant line bundle associated to the Θ-divisor of sheaves with a section.

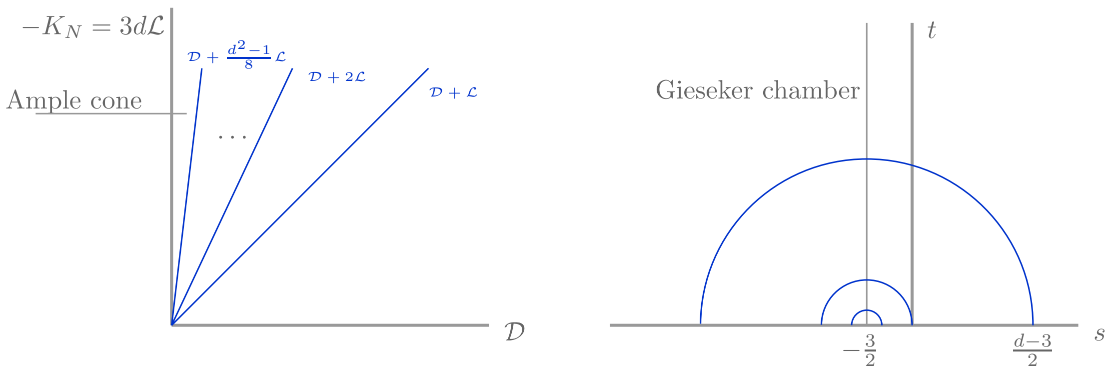

Figure 4.1. Mori and Bridgeland chambers.

4.2 Duality

Let X be a smooth projective surface, $D, H \in \mathrm{Num}(X)_{\mathbb{R}}$ with H ample. Let $\sigma_{D,tH} = (Z_{D,tH}, \mathcal{A}_{DH})$ be the stability condition of Theorem 2.12, and assume that projective coarse moduli spaces for $\sigma_{D,tH}$ and $\sigma_{-D+K,tH}$ are known to exist. For example, X can be $\mathbb{P}^2$ [ABCH13] or a $K3$ surface [BM14]. This chapter is devoted to prove

Theorem 4.4. *The functor $\mathcal{F} \mapsto \mathcal{F}^D := R\mathcal{H}om(\mathcal{F}, \omega_X)[1]$ induces an isomorphism between the Bridgeland moduli $M_{D,tH}(ch(F)) \cong M_{-D+K_X,tH}(ch(F^D))$ provided that $ch(F)$ is the chern character of an object in $\mathcal{A}_{DH}$ of phase in $(0,1)$.*

This result was proven by Maican [Mai10] for the Gieseker moduli of sheaves on $\mathbb{P}^n$ supported on curves. The theorem above recovers Maican's for $X = \mathbb{P}^2$ and $t \gg 0$. The proof in this context is identical to Maican's original proof modulo the following

Lemma 4.5. *Let E be a $\sigma_{D,tH}$-(semi)stable object in $\mathcal{P}(0,1)$. Then*

(a) If E is stable, then it is quasi-isomorphic to a two-term complex of vector bundles $E^{-1} \to E^0$.

(b) $\mathcal{H}^{-1}(E)$ is torsion-free with semistable factors of slope $< DH$.

(c) If $A \in \mathcal{A}_{DH}$ is an object all of whose semistable factors belong to $\mathcal{P}(0,1)$, then $A^D \in \mathcal{A}_{(-D+K_X)H}$.

(d) $E^D \in \mathcal{A}_{(-D+K_X)H}$ *is* $\sigma_{-D+K_X,tH}$*-(semi)stable.*

(e) If $E, F \in \mathcal{A}_{DH}$ *are S-equivalent, then so are* $E^D, F^D \in \mathcal{A}_{(-D+K_X)H}$.

(f) For any flat family $\mathbf{F} \in D^b(S \times X)$ *with fibers of invariants* $ch(F_s)$*, there is a flat family* $\mathbf{F}^D \in D^b(S \times X)$ *with fibers of invariants* $ch({F_s}^D)$ *such that*

$$Li_s^*(\mathbf{F}^D) \cong (Li_s^*\mathbf{F})^D.$$

Proof. Part (a) is the content of Remark 2.16. For (b) notice that if $\mu_{max}(\mathcal{H}^{-1}(E)) = DH$ then $\mathcal{H}^{-1}(E)[1]$ will have a subobject of phase 1 destabilizing E. Assume that E is stable. To prove that $E^D \in \mathcal{A}_{(-D+K)H}$ note that for any coherent sheaf F with semistable factors of slope $< DH$ (resp. $> DH$), we have

$$\mathcal{H}^i(F^D) = \begin{cases} \text{torsion free sheaf with } \mu_{min} > -D\cdot H + K\cdot H & \\ (\text{resp. } \mu_{max} < -D\cdot H + K\cdot H) & \text{if } i = -1 \\ \text{torsion or } 0 & \text{if } i = 0 \\ 0\text{-dim torsion sheaf or } 0 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(which can be proven by taking a minimal free resolution for F). Taking cohomology on the short exact sequence

$$0 \to \mathcal{H}^0(E)^D \to E^D \to \mathcal{H}^{-1}(E)[1]^D \to 0$$

we get the long exact sequence of sheaves

$$\begin{gathered} 0 \to \mathcal{H}^{-1}(\mathcal{H}^0(E)^D) \to \mathcal{H}^{-1}(E^D) \to \mathcal{H}^{-1}(\mathcal{H}^{-1}(E)[1]^D) \to \\ \to \mathcal{H}^0(\mathcal{H}^0(E)^D) \to \mathcal{H}^0(E^D) \to \mathcal{H}^0(\mathcal{H}^{-1}(E)[1]^D) \to \mathcal{H}^1(\mathcal{H}^0(E)^D) \to 0 \end{gathered}$$

since by (a) E^D is a two-term complex of vector bundles. But $\mathcal{H}^{-1}(E)[1]^D = \mathcal{H}^{-1}(E)^D[-1]$, and so

$$\mathcal{H}^{-1}(\mathcal{H}^{-1}(E)[1]^D) = 0.$$

This implies that $\mathcal{H}^{-1}(E^D) \in \mathcal{F}_{(-D+K)H}$, $\mathcal{H}^0(\mathcal{H}^0(E)^D)$ is the torsion subsheaf of $\mathcal{H}^0(E^D)$ and because $\mathcal{H}^1(\mathcal{H}^0(E)^D)$ is a zero-dimensional sheaf we have $\mathcal{H}^0(E^D) \in \mathcal{Q}_{(-D+K)H}$.

Moreover, this proves that if $A \in \mathcal{P}(0,1)$ is stable then A^D is an element of $\mathcal{A}_{(-D+K)H}$. For arbitrary $A \in \mathcal{P}(0,1)$, A is in the extension closure of some stable objects $A_1, \ldots, A_k \in \mathcal{A}_{DH}$ of the same phase, and so

$$A^D \in \langle {A_1}^D, \ldots, {A_k}^D \rangle \subset \mathcal{A}_{(-D+K)H}.$$

By the same argument we get (c).

Assume for the moment that E is stable, then there is no injective map

$$0 \to \mathcal{K} \to E^D$$

in $\mathcal{A}_{(-D+K)H}$ with $\mathcal{K}$ having at least one of its semistable factors of phase 1. If so, there would be an inclusion

$$0 \to A \to E^D$$

with $A \in \mathcal{P}(1)$ stable; i.e., $A = \mathbb{C}_x$ or $A = F[1]$ for some locally free sheaf F with μ_H-semistable factors of slope $(-D+K)H$ ([Bri08, 10.1(b)]). But if E is stable, then E^D is derived equivalent to a two-term complex of vector bundles implying $\mathrm{Hom}(\mathbb{C}_x, E^D) = 0$. Also

$$\mathrm{Hom}(F[1], E^D) = \mathrm{Hom}(F, \mathcal{H}^{-1}(E^D)) = 0$$

in virtue of Schur's lemma.

This allows us to conclude that E^D is stable; indeed, if there is a destabilizing sequence

$$0 \to A \to E^D \to B \to 0,$$

we can choose B to be stable, and by the argument above we know that all semistable factors of A have phase in $(0,1)$. Then by dualizing this sequence, we get a destabilizing sequence for E in $\mathcal{A}_{DH}$ since

$$\mu_{D,tH}(\cdot) = -\mu_{-D+K,tH}(\cdot)^D.$$

We conclude that E^D is semistable for all semistable objects E of phase in $(0,1)$ just by dualizing the Jordan-Holder filtration of E.

Let $0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$ be a Jordan-Holder filtration for E in $\mathcal{A}_{DH}$, then $(E/F_n)^D \subset (E/F_{n-1})^D \subset \cdots \subset E^D$ is a Jordan-Holder filtration for E^D in $\mathcal{A}_{(-D+K)H}$ with stable factors $(F_i/F_{i-1})^D$. This also gives part (d).

For the last part let $\mathbf{F}^D := R\mathcal{H}om(\mathbf{F}, \omega_{S\times X/S})$, then

$$Li_s^*(R\mathcal{H}om(\mathbf{F}, \omega_{S\times X/S})) \cong R\mathcal{H}om(Li_s^*\mathbf{F}, \omega_X) \in \mathcal{A}_{(-D+K)H}.$$

$\square$

Proof of Theorem 4.4. Every flat family $\mathbf{F} \in D^b(S \times X)$ gives a morphism $\pi : S \to M_{D,tH}(v)$ and a morphism $\pi^D : S \to M_{-D+K,tH}(v^D)$ corresponding to the family $\mathbf{F}^D$ of the lemma. By part (e), π^D is constant on the fibers of π. Then π^D factors through a morphism $M_{D,tH}(v) \to M_{-D+K,tH}(v^D)$, which sends the closed point representing E to the closed point representing E^D. The symmetry of the situation and the fact that $(\_)^{DD} = Id$ prove that such morphism is an isomorphism. $\square$

Remark 4.6. In the special case when $X = \mathbb{P}^2$ and $v = (0, d, -3d/2)$ duality gives an automorphism $(\_)^D : M_{-3/2,t}(v) \cong M_{-3/2,t}(v)$ for all $t > 0$.

Corollary 4.7. *Let $N([C], \chi)$ denote the moduli space of Gieseker semistable sheaves of Euler-Poincare characteristic χ supported on a curve of class $[C]$. The functor $\mathcal{F} \mapsto \mathscr{E}xt^1(\mathcal{F}, \omega_X)$ induces an isomorphism between $N([C], \chi)$ and $N([C], -\chi)$.*

Proof. Take $D = K/2$ in the duality theorem. If $r(E) = 0$ and $c_1(E) = [C]$, then

$$\mu_{K/2,tH}(E) = \frac{\chi(E)}{tC \cdot H},$$

and therefore a sheaf of those invariants that is $\sigma_{K/2,tH}$ semistable is also Gieseker semistable. By Theorem 2.22 we know that the values of t for which there is an inclusion of objects $A \hookrightarrow E$ with $\mu_{K/2,tH}(A) = \mu_{K/2,tH}(E)$ is bounded above (this also follows by a result of Maciocia [Mac12, Theorem 3.11] when considering the family of stability conditions $\sigma_{K/2+sH,tH}$). If E is an object that is $\sigma_{K/2,tH}$-semistable for all $t \gg 0$, then E must be a sheaf since otherwise $\mathcal{H}^{-1}(E)[1]$ would destabilize E. If $A \to E$ is an inclusion in $\mathcal{A}_{KH/2}$, then A must be a sheaf, and if it destabilizes E, it must be a sheaf of positive rank. But a simple computation shows that for $t \gg 0$ one has

$$\mu_{K/2,tH}(A) < \mu_{K/2,tH}(E),$$

and so the inclusion $A \to E$ must produce a wall. Since the walls are bounded above we conclude that above all walls E is $\sigma_{K/2,tH}$-semistable. The coarse moduli spaces $N([C],\chi)$ were constructed by C. Simpson [Sim94] via invariant theory, then the conclusion follows from the duality theorem. $\square$

Remark 4.8. We remark that Corollary 4.7 was already known not only for $\mathbb{P}^2$ [Mai10] but also for elliptic surfaces by Yoshioka [Yos01] and by Arbarello, Saccà and Ferreti for special kinds of K3 surfaces [ASF12].

4.3 An embedded problem: flips of secant varieties

In [Ver01] and [Ver02] P. Vermeire describes a sequence of flips for the secant varieties of an embedding $X \hookrightarrow \mathbb{P}^N$ of an algebraic surface. This sequence of flips is constructed in similar fashion to the flips obtained by Thaddeus [Tha94] when studying variation of GIT for moduli spaces of stable pairs on curves. The first of these flips is easy to describe and it is the content of [Ver01, 4.13]. Roughly speaking, if the embedding of X is sufficiently ample such that it can be generated by quadrics with only linear syzygies, then there is a flip diagram

$$\begin{tikzcd}
 & \tilde{M} \arrow[dl, "\pi"'] \arrow[dr, "h"] & \\
\mathrm{bl}_X(\mathbb{P}^N) \arrow[d] \arrow[dr, "\varphi^+"] & & M \arrow[dl, "\varphi^-"] \\
\mathbb{P}^N \arrow[r, dotted, "\varphi"'] & \mathbb{P}^s &
\end{tikzcd}$$

where $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^s$ is the rational map given by the forms defining X and $\tilde{M}$ is the blow-up of $\mathrm{bl}_X(\mathbb{P}^N)$ along the strict transform of the secant variety $\widetilde{SecX}$. The diagram restricts to

$$\begin{tikzcd}
 & E \arrow[dl, "\pi"'] \arrow[dr, "h"] & \\
\mathbb{P}(\mathcal{E}) \arrow[dr, "\varphi^+"'] & & \mathbb{P}(\mathcal{F}) \arrow[dl, "\varphi^-"] \\
 & Hilb^2(X) &
\end{tikzcd}$$

where $\mathbb{P}(\mathcal{E}) \cong \widetilde{SecX}$ and $\mathcal{F} = {\varphi^+}_*(N^*_{\mathbb{P}(\mathcal{E})/\,\mathrm{bl}_X(\mathbb{P}^N)} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))$.

We will see that in the case $X = \mathbb{P}^2$ such flips appear naturally when running the MMP for the Gieseker moduli $M(0, d, -\frac{3d}{2})$ for d odd (or $M(0, d, -d)$ for d even).

Consider the exceptional loci for the first flip of $M(0, d, -\frac{3d}{2})$ ($d \geq 5$ odd):

$$
\begin{aligned}
E_0^+ &: \; 0 \to \mathcal{O}\left((d-3)/2\right) \to F \to \mathcal{O}\left((-d-3)/2\right)[1] \to 0 \\
E_0^- &: \; 0 \to \mathcal{O}\left((-d-3)/2\right)[1] \to G^\bullet \to \mathcal{O}\left((d-3)/2\right) \to 0
\end{aligned}
$$

these are obtained from the set-theoretic wall-crossing since $\mathcal{O}\left((d-3)/2\right)$ and $\mathcal{O}\left((-d-3)/2\right)[1]$ are stable for every value of t. E_0^+ and E_0^- are projective spaces, indeed:

$$
\begin{aligned}
E_0^+ &\cong \mathbb{P}\left(\mathrm{Ext}^1(\mathcal{O}\left((-d-3)/2\right)[1], \mathcal{O}\left((d-3)/2\right))\right), \\
E_0^- &\cong \mathbb{P}\left(\mathrm{Ext}^1(\mathcal{O}\left((d-3)/2\right), \mathcal{O}\left((-d-3)/2\right)[1])\right) \\
&= \mathbb{P}(H^2(\mathbb{P}^2, \mathcal{O}(-d))) \\
&\cong \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(d-3))^\vee).
\end{aligned}
$$

There is a natural $\mathbb{P}^2$ embedded in E_0^- by the complete linear series $|\mathcal{O}(d-3)|$. This Veronese surface can be described in terms of extensions, it is the set of complexes $G^\bullet$ fitting into a commutative diagram

$$
\begin{tikzcd}
 & & \mathcal{I}_p\left((d-3)/2\right) \arrow[dl] \arrow[d, hook] \\
\mathcal{O}\left((-d-3)/2\right)[1] \arrow[r, hook] \arrow[d, equal] & G_p^\bullet \arrow[r, two heads] \arrow[d] & \mathcal{O}\left((d-3)/2\right) \arrow[d] \\
\mathcal{O}\left((-d-3)/2\right)[1] \arrow[r, hook] & \mathcal{G} \arrow[r, two heads] & \mathbb{C}_p
\end{tikzcd}
\tag{4.5}
$$

Note that $\mathcal{G}$ is unique (up to scalars) since $\mathrm{ext}^1(\mathbb{C}_p, \mathcal{O}\left((-d-3)/2\right)[1]) = 1$. Thus $G_p^\bullet$ is the image under the pullback homomorphism

$$
\mathrm{Ext}^1(\mathbb{C}_p, \mathcal{O}\left((-d-3)/2\right)[1]) \hookrightarrow \mathrm{Ext}^1(\mathcal{O}\left((d-3)/2\right), \mathcal{O}\left((-d-3)/2\right)[1]).
$$

But we know that

$$
\begin{aligned}
\operatorname{Ext}^1(\mathbb{C}_p, \mathcal{O}\left((-d-3)/2\right)[1]) &\cong \operatorname{Ext}^1((\mathcal{O}\left((-d-3)/2\right)[1])^D, (\mathbb{C}_p)^D) \\
&= \operatorname{Ext}^1(\mathcal{O}\left((d-3)/2\right), (\mathbb{C}_p)^D),
\end{aligned}
$$

so $G_p^\bullet$ is also the image under the push-forward map

$$
\operatorname{Ext}^1(\mathcal{O}\left((d-3)/2\right), (\mathbb{C}_p)^D) \hookrightarrow \operatorname{Ext}^1(\mathcal{O}\left((d-3)/2\right), \mathcal{O}\left((-d-3)/2\right)[1]).
$$

Applying the functor $(\_)^D$ to the pullback diagram above gives us the push-forward diagram

$$
\begin{tikzcd}
\mathbb{C}_p^D \arrow[r, hook] \arrow[d, hook] & \mathcal{G}^D \arrow[r, two heads] \arrow[d] & \mathcal{O}\left((d-3)/2\right) \arrow[d, equal] \\
\mathcal{O}\left((-d-3)/2\right)[1] \arrow[r, hook] \arrow[d] & (G_p^\bullet)^D \arrow[r, two heads] \arrow[dl] & \mathcal{O}\left((d-3)/2\right) \\
\left(\mathcal{I}_p((d-3)/2)\right)^D & &
\end{tikzcd}
$$

Proposition 4.9. *The elements of E_0^- are fixed by the duality automorphism.*

Proof. From the discussion above we know that $G_p^\bullet = (G_p^\bullet)^D$, and so the duality automorphism that restricts to an automorphism $(\_)^D|_{E_0^-} : E_0^- \to E_0^-$ fixes the $(d-3)$-uple embedding of $\mathbb{P}^2$. Since every automorphism of $E_0^- \cong \mathbb{P}^N$ is linear, then $(\_)^D|_{E_0^-}$ must be the identity. $\square$

From now on we denote by X the $(d-3)$-uple embedding of $\mathbb{P}^2$ inside E_0^-. The exceptional loci for the second flip are

$$
\begin{aligned}
E_1^+ &: \quad 0 \to \mathcal{I}_p((d-3)/2) \to F \to \mathcal{I}_q^\vee((-d-3)/2)[1] \to 0; \quad p, q \in \mathbb{P}^2 \\
E_1^- &: \quad 0 \to \mathcal{I}_q^\vee((-d-3)/2)[1] \to G \to \mathcal{I}_p((d-3)/2) \to 0; \quad p, q \in \mathbb{P}^2.
\end{aligned}
$$

Since the Bridgeland moduli for the Hilbert scheme of 1 point is constant (equal to $\mathbb{P}^2$), then the description of E_1^- above is given by Corollary 2.36.

Proposition 4.10. *(a) E_1^+ and E_1^- are both projective bundles over $\mathbb{P}^2 \times \mathbb{P}^2$.*

(b) $E_1^+ \cap E_0^- = X$.

(c) The closure of $E_0^- \setminus X$ in M_1^- is isomorphic to the blow-up of E_0^- along X.

Proof. For part (a) one only needs to verify that

$$\begin{aligned}
&\mathrm{ext}^1(\mathcal{I}_q^\vee((-d-3)/2)[1], \mathcal{I}_p((d-3)/2)), \text{ and} \\
&\mathrm{ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1])
\end{aligned}$$

are constant, and the rest of the argument follows as in [AB13, Proposition 4.2]. We have

$$\begin{aligned}
\mathrm{Ext}^1(\mathcal{I}_q^\vee((-d-3)/2)[1], \mathcal{I}_p((d-3)/2)) &= \mathrm{Hom}(\mathcal{O}, \mathcal{I}_\mathrm{p} \otimes \mathcal{I}_\mathrm{q}(\mathrm{d})), \\
\mathrm{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1]) &= \mathrm{Ext}^2(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1]) \\
&\cong \mathrm{Hom}(\mathcal{O}, \mathcal{I}_\mathrm{p} \otimes \mathcal{I}_\mathrm{q}(\mathrm{d}-3))^\vee.
\end{aligned}$$

Note that we can use ordinary tensor instead of derived tensor. This is because ideal sheaves have a two-term resolution by locally free sheaves. For $p \neq q$ there is no problem. For $p = q$ one gets constant dimension because

$$H^1(\mathbb{P}^2, \mathcal{I}_p^2(k)) = 0 \text{ for } k > 0,$$

which follows for example by Bertram-Ein-Lazarsfeld vanishing:

Theorem 4.11 (Bertram-Ein-Lazarsfeld, [BEL91])**.** *Assume that $X \subset \mathbb{P}^r$ is (scheme-theoretically) cut out by hypersurfaces of degrees $d_1 \geq d_2 \geq \cdots \geq d_m$. Then*

$$H^i(\mathbb{P}^r, \mathcal{I}_X^a(k)) = 0 \quad \textit{for all} \quad i \geq 1$$

provided that $k \geq ad_1 + d_2 + \cdots + d_e - r$, where $e = codim(X, \mathbb{P}^r)$.

For part (b) diagram (4.5) already shows that $X \subset E_1^+ \cap E_0^-$. For the other inclusion one notices that

$$\begin{aligned}
&\mathrm{hom}(\mathcal{I}_p((d-3)/2), \mathcal{O}((d-3)/2)) = 1, \text{ and} \\
&\mathrm{hom}(\mathcal{I}_p((d-3)/2), \mathcal{O}((-d-3)/2)[1]) = 0.
\end{aligned}$$

More can be said, since E_0^- is fixed by the duality automorphism then E_1^+ intersects E_0^- along a section over the diagonal $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$. Since the morphism π_1^+ :

$M_1^+ \to M_1$ collapses the fibers of E_1^+ then $\pi_1|_{E_0^-} : E_0^- \to M_1$ is a closed immersion. By Lemma 4.12 we have a diagram

$$\begin{tikzcd}
& \mathrm{bl}_{E_1^+} M_1^+ \arrow[d] \arrow[dr] & \\
\mathrm{bl}_X E_0^- \arrow[ur] \arrow[d] & M_1^+ \arrow[dr] & M_1^- \arrow[d] \\
E_0^- \arrow[ur, hook] \arrow[rr, hook, "\pi_1^+|_{E_0^-}"'] & & M_1
\end{tikzcd}$$

which proves $\mathrm{bl}_X E_0^- \subset M_1^-$ completing the proof of part (c). $\square$

Lemma 4.12. *The fiber product* $M_1^+ \times_{M_1} M_1^-$ *is isomorphic to the common blow-up* $\mathrm{bl}_{E_1^+} M_1^+ = \mathrm{bl}_{E_1^-} M_1^-$.

Proof. A proof of this statement was already given in [BMW14] for the case $d = 5$, which generalizes for all d (odd) without change. One notices the following vanishing of Ext groups:

$$\begin{aligned}
\mathrm{Hom}(\mathcal{I}_\mathrm{p}((\mathrm{d}-3)/2), \mathcal{I}_\mathrm{q}^\vee((-\mathrm{d}-3)/2)[1]) &= 0,\\
\mathrm{Ext}^2(\mathcal{I}_q^\vee((-d-3)/2)[1], \mathcal{I}_q^\vee((-d-3)/2)[1]) &= 0,\\
\mathrm{Ext}^2(\mathcal{I}_p((d-3)/2), \mathcal{I}_p((d-3)/2)) &= 0,\\
\mathrm{Ext}^2(F, \mathcal{I}_p((d-3)/2)) &= 0,\\
\mathrm{Ext}^2(\mathcal{I}_q^\vee((-d-3)/2)[1], F) &= 0,
\end{aligned}$$

for every $p, q \in \mathbb{P}^2$ and $F \in E_1^+$. The first is obvious when $p \neq q$, for $p = q$ one uses Serre duality and Bertram-Ein-Lazarsfeld vanishing.

The last two are obtained by using Serre duality and the fact that E is Bridgeland stable.

Then by applying the functors $\mathrm{Hom}(F, \cdot\,)$ and $\mathrm{Hom}(\,\cdot\,, F)$ to the exact sequence

$$0 \to \mathcal{I}_p((d-3)/2) \to F \to \mathcal{I}_q^\vee((-d-3)/2)[1] \to 0$$

one gets diagrams

$$
\begin{tikzcd}
 & & 0 \arrow[d] \\
 & & \mathrm{Ext}^1(\mathcal{I}_q^\vee((-d-3)/2)[1], \mathcal{I}_q^\vee((-d-3)/2)[1]) \arrow[d] \\
\mathrm{Ext}^1(F, \mathcal{I}_p((d-3)/2)) \arrow[r, hook] & \mathrm{Ext}^1(F,F) \arrow[r, two heads] \arrow[dr, "f"] & \mathrm{Ext}^1(F, \mathcal{I}_q^\vee((-d-3)/2)[1]) \arrow[d] \\
 & & \mathrm{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1]) \arrow[d] \\
 & & 0
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & & 0 \arrow[d] \\
 & & \mathrm{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_p((d-3)/2)) \arrow[d] \\
\mathrm{Ext}^1(\mathcal{I}_q^\vee((-d-3)/2)[1], F) \arrow[r, hook] & \mathrm{Ext}^1(F,F) \arrow[r, two heads] \arrow[dr, "f"] & \mathrm{Ext}^1(\mathcal{I}_p((d-3)/2), F) \arrow[d] \\
 & & \mathrm{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1]) \arrow[d] \\
 & & 0
\end{tikzcd}
$$

Then there is a induced map

$$\mathrm{Ext}^1(\mathcal{I}_q^\vee((-d-3)/2)[1], \mathcal{I}_p((d-3)/2)) \to \ker f$$

with kernel $\mathbb{C}$ and whose cokernel can be identified with

$\mathrm{Ext}^1(\mathcal{I}_q^\vee((-d-3)/2)[1], \mathcal{I}_q^\vee((-d-3)/2)[1]) \oplus \mathrm{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_p((d-3)/2))$,

and we get an exact sequence

$$0 \to \ker f \to \mathrm{Ext}^1(F,F) \to \mathrm{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1]) \to 0.$$

Thus $\ker f$ can be identified with the tangent space of E_1^+ at the point $[F]$. We get

$$0 \to (\mathbb{T}E_1^+)_{[F]} \to (\mathbb{T}M_1^+|_{E_1^+})_{[F]} \to \mathrm{Ext}^1(\mathcal{I}_p((d-3)/2), \mathcal{I}_q^\vee((-d-3)/2)[1]) \to 0,$$

and therefore an exact sequence of sheaves

$$0 \to \mathbb{T}E_1^+ \to \mathbb{T}M_1^+|_{E_1^+} \to (\pi_1^+|_{E_1^+})^*((\pi_1^-|_{E_1^-})_*\mathcal{O}_{E_1^-}(1)) \to 0.$$

Similarly one gets

$$0 \to \mathbb{T}E_1^- \to \mathbb{T}M_1^-|_{E_1^-} \to (\pi_1^-|_{E_1^-})^*((\pi_1^+|_{E_1^+})_*\mathcal{O}_{E_1^+}(1)) \to 0.$$

This proves that we have a fiber square

$$\begin{tikzcd}
\mathbb{P}(N_{E_1^+/M_1^+}) \cong \mathbb{P}(N_{E_1^-/M_1^-}) \arrow[r] \arrow[d] & E_1^- \arrow[d, "\pi_1^-|_{E_1^-}"] \\
E_1^+ \arrow[r, "\pi_1^+|_{E_1^+}"'] & \mathbb{P}^2 \times \mathbb{P}^2
\end{tikzcd}$$

which completes the proof. $\square$

We now study the third flip for $d \geq 7$ odd. Since $d \geq 7$ then ideal sheaves of length two zero-subschemes are Bridgeland stable at the wall, and since $2 < \frac{d-1}{2}$ then there is unique solution for the inequality 4.4, and so the exceptional loci are:

$$\begin{aligned}
E_2^+ \quad &: \quad 0 \to \mathcal{I}_Z((d-3)/2) \to F \to \mathcal{I}_W^\vee((-d-3)/2)[1] \to 0 \quad |Z| = |W| = 2 \\
E_2^- \quad &: \quad 0 \to \mathcal{I}_W^\vee((-d-3)/2)[1] \to G \to \mathcal{I}_Z((d-3)/2) \to 0 \quad |Z| = |W| = 2.
\end{aligned}$$

Again, Bertram-Ein-Lazarsfeld vanishing exposes E_2^+ and E_2^- as projective bundles over $\mathrm{Hilb}^2(\mathbb{P}^2) \times \mathrm{Hilb}^2(\mathbb{P}^2)$.

Our plan is to study the restriction of the directed MMP for $M(0, d, -3d/2)$ to E_0^-. It is convenient to fix some notation. Inductively define $Y_1^+ := E_0^-$, Y_i^- is the closure of the image of Y_i^+ by the rational map

$$M_i^+ \dashrightarrow M_i^-$$

and $Y_{i+1}^+ := Y_i^-$. Then, for instance, $Y_1^- = Y_2^+ = \mathrm{bl}_X E_0^-$.

Proposition 4.13. *E_2^+ intersects Y_2^+ along the strict transform of the secant variety $\widetilde{SecX}$ which is a projective bundle over* $\mathrm{Hilb}^2(\mathbb{P}^2)$.

Proof. The computation is very similar to the one we did when computing $E_1^+ \cap E_0^-$. Let $Z = p + q$ where $p, q \in \mathbb{P}^2$ and $p \neq q$.

We have a pullback diagram

$$\begin{tikzcd}
 & & \mathcal{I}_Z\left((d-3)/2\right) \arrow[dl] \arrow[d, hook] & \\
\mathcal{O}\left((-d-3)/2\right)[1] \arrow[r, hook] \arrow[d, equal] & G_Z^\bullet \arrow[r, two heads] \arrow[d] & \mathcal{O}\left((d-3)/2\right) \arrow[d] & (4.6) \\
\mathcal{O}\left((-d-3)/2\right)[1] \arrow[r, hook] & \mathcal{G} \arrow[r, two heads] & \mathbb{C}_Z &
\end{tikzcd}$$

The difference here is that $\mathrm{ext}^1(\mathbb{C}_Z, \mathcal{O}((-d-3)/2)[1]) = 2$, which corresponds to the line passing through p and q removing p and q. Thus intersection of $E_2^+ \setminus E_1^-$ with $E_0^- \setminus X$ is $SecX \setminus X$, which proves the first claim.

The problem arises when considering $Z = 2p$. In this case,

$$\mathrm{ext}^1(\mathbb{C}_Z, \mathcal{O}((-d-3)/2)[1]) = 3.$$

But since in M_2^+ we already flipped E_1^+ then not all the complexes G_Z obtained this way are Bridgeland stable. Instead, the complexes G_{2p} fitting into a commutative diagram

$$\begin{tikzcd}
 & & \mathcal{I}_{2p}\left((d-3)/2\right) \arrow[dl] \arrow[d, hook] & \\
\mathcal{I}_p((-d-3)/2)[1] \arrow[r, hook] \arrow[d, equal] & G_{2p} \arrow[r, two heads] \arrow[d] & \mathcal{I}_p((d-3)/2) \arrow[d] & (4.7) \\
\mathcal{I}_p((-d-3)/2)[1] \arrow[r, hook] & \mathcal{G} \arrow[r, two heads] & \mathbb{C}_p &
\end{tikzcd}$$

are Bridgeland stable. The objects G_{2p} form the fiber of $\widetilde{SecX}$ over $Z = 2p$. $\square$

Lemma 4.14. *The fiber product $M_2^+ \times_{M_2} M_2^-$ is isomorphic to the common blow-up $\mathrm{bl}_{E_2^+} M_2^+ = \mathrm{bl}_{E_2^-} M_2^-$.*

Proof. The proof is similar to the proof of Lemma 4.12. The right vanishing is again a consequence of Bertram-Ein-Lazarsfeld vanishing. $\square$

This completes Vermeire's first flip since by restricting the fiber diagram of Lemma 4.14 one gets

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d] & \mathrm{bl}_{\widetilde{SecX}}(\mathrm{bl}_X E_0^-) \arrow[r, hook] \arrow[d] & M_2^+ \times_{M_2} M_2^- \arrow[r] \arrow[d] & M_2^- \arrow[d] \\
\widetilde{SecX} \arrow[r, hook] \arrow[drrr] & \mathrm{bl}_X(E_0^-) \arrow[r, hook] & M_2^+ \arrow[r] & M_2 \\
 & & & Hilb^2(\mathbb{P}^2) \arrow[u, hook]
\end{tikzcd}$$

Remark 4.15. In [Ver01] it is mentioned that flips of secant varieties are closely related to the geometry of $\mathrm{Hilb^n(X)}$. By Proposition 4.3 and Corollary 2.36 and the results of this section, one sees that indeed flips of secant varieties of Veronese surfaces are related to the birational geometry of $Hilb^n(\mathbb{P}^2)$.

By using diagrams similar to 4.5 and 4.6 one sees that every rank-1 wall produces a birational transformation of E_0^- whose exceptional locus contains the strict transform of some higher secant variety of X. Indeed, for $\ell < (d-1)/2$ the exceptional locus for the induced birational transformation of E_0^-, corresponding to crossing the wall W_ℓ, is the strict transform of $Sec^{\ell-1}X$. For $\ell \geq (d-1)/2$ the exceptional locus is reducible, and the middle component $E_{\ell,0}$ intersects E_0^- along the strict transform of $Sec^{\ell-1}X$. The intersection $E_{\ell,-i} \cap E_{\ell,0} \cap E_0^-$ is the locus in $\widetilde{Sec^{\ell-1}}X$ of $(\ell-1)$-dimensional planes passing through ℓ different points, $i(d-i)/2$ of them lying on the image by the $(d-3)$-uple embedding of a curve $C \subset \mathbb{P}^2$ of degree i. Since E_0^- is fixed by the duality automorphism, this completely describes the loci for the induced MMP on E_0^-.

4.3.1 The divisorial contraction

We want to study what happens to our restricted MMP when crossing the wall W_Θ corresponding to the theta divisor (i.e., the closure of the set of those sheaves that admit at least one nonzero section). The theta divisor is fixed by the duality automorphism, and therefore it corresponds to extensions of the form

$$0 \to N \to F \to \mathcal{O}(-3)[1] \to 0,$$

where N is an element in the corresponding model $\mathcal{N}$ of $Hilb^n(\mathbb{P}^2) \otimes \mathcal{O}(d-3)$ of $n = d(d-3)/2$ points on the plane.

Remark 4.16. One can originally think of the dual extensions, but this version allows us to compute the intersection with the first flipped locus more effectively.

The intersection of the divisor Θ with E_0^- corresponds to the extensions F fitting into the push-forward diagrams

$$\begin{tikzcd}
\mathcal{O}_C(-3) \arrow[r, hook] \arrow[d, hook] & G \arrow[r, two heads] \arrow[d] & \mathcal{O}((d-3)/2) \arrow[d, equal] \\
\mathcal{O}((-d-3)/2)[1] \arrow[r, hook] \arrow[d, two heads] & F \arrow[r, two heads] \arrow[dl, two heads] & \mathcal{O}((d-3)/2) \\
\mathcal{O}(-3)[1] & &
\end{tikzcd}$$

where $C \subset \mathbb{P}^2$ is a curve of degree $(d-3)/2$. Indeed, the middle vertical sequence of arrows is exact, and G corresponds to those complexes produced when flipping the locus in $Hilb^n(\mathbb{P}^2)$ of n points on a curve of degree $(d-3)/2$. This intersection is therefore a projective bundle over the Hilbert scheme of plane curves of degree $(d-3)/2$. An example of this situation was already observed in [BMW14] for the case $d = 5$ where the intersection of Θ with E_0^- was exactly the strict transform of the secant variety of the Veronese surface in $\mathbb{P}^5$.

Notice that this is intersection is not exactly what gets contracted when crossing W_Θ since after several flips we may have replaced some of these objects by new ones. What we know is that the object G above must have $\mathcal{O}$ as a subobject, and therefore crossing W_Θ must introduce objects E, fitting into an exact sequence

$$0 \to \mathcal{O}(-3)[1] \to E \to \mathcal{O} \oplus \mathcal{G} \to 0,$$

where $ch(\mathcal{G}) = (0, d-3, -3(d-3)/2)$. Thus these new objects E are all strictly semistable, in fact pseudo-stable. Further analysis tells us that if $\mathcal{G}$ is a sheaf, then it must fit into an exact sequence $0 \to \mathcal{O}_C(-3) \to \mathcal{G} \to \mathcal{O}_C((d-3)/2) \to 0$.

Semistability of $\mathcal{G}$ at W_Θ says that if $\mathcal{G}$ is a complex, then at least it has to fit into an exact sequence of the form

$$0 \to A \to \mathcal{G} \to A^D \to 0,$$

where A is a semistable object of invariants $ch(A) = \left(0, \frac{d-3}{2}, -\frac{(d-3)(d+9)}{8}\right)$.

4.3.2 The last birational model

One could ask what is the moduli space we obtain after the divisorial contraction and what is the strict transform of ${E_0}^-$. The answer to the first question comes from the identification with the quiver moduli. Values of $(-3/2, t)$ near W_C are all inside the quiver region corresponding to $k = -1$. In this case we can compute the dimension invariants

$$\begin{bmatrix} \frac{k(k-1)}{2} & \frac{-(2k-1)}{2} & 1 \\ k(k-2) & -(2k-2) & 2 \\ \frac{(k-1)(k-2)}{2} & \frac{-(2k-3)}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ d \\ -3d/2 \end{bmatrix} = \begin{bmatrix} 0 \\ d \\ d \end{bmatrix}$$

A more detailed analysis shows that above W_C, the polarization $(-\theta, \theta)$ satisfies $\theta > 0$, at W_C we have $\theta = 0$, and below W_C it satisfies $\theta < 0$. Therefore the last model corresponds to the moduli space $N(3, d, d)$ studied in [DM11] of semistable morphisms

$$W\mathcal{O}(-2) \to W^*\mathcal{O}(-1),$$

where $\dim W = d$. The moduli space at W_C is just a point and below W_C is empty, which proves our assertion that W_C was the collapsing wall.

In order to understand what is going to be the last birational model of E_0^-, let us take a look at the simplest but yet interesting $M_H(0, 3, -9/2)$ studied by Le Potier. Le Potier showed that $M_H(0, 3, -9/2)$ is the blow up of $N(3, 3, 3)$ at the complement of the dense open subset of injective morphisms $3\mathcal{O}(-2) \hookrightarrow 3\mathcal{O}(-1)$. This complement consists of a single orbit, which is the orbit corresponding to the skew matrix

$$\begin{pmatrix} 0 & -z & x \\ z & 0 & -y \\ -x & y & 0 \end{pmatrix}$$

$$\begin{tikzcd}
\mathcal{O}(-3) \arrow[dr] & & & & \mathcal{O} \\
& 3\mathcal{O}(-2) \arrow[rr, "\begin{pmatrix} 0 & -z & x \\ z & 0 & -y \\ -x & y & 0 \end{pmatrix}"] \arrow[dr] & & 3\mathcal{O}(-1) \arrow[ur] & \\
& & \Omega^1 \arrow[ur] & &
\end{tikzcd}$$

As in the example above the general skew-map $W\mathcal{O}(-2) \to W^*\mathcal{O}(-1)$ will drop rank by 1 everywhere, and therefore it must have a kernel and a cokernel that are line bundles; indeed, as a complex it should fit into an exact sequence

$$0 \to \mathcal{O}((-d-3)/2)[1] \to E \to \mathcal{O}((d-3)/2) \to 0.$$

On the other hand, by Proposition 4.2 all the complexes in E_0^- are stable rather than pseudo-stable. Any stable complex in the last model for E_0^- must be, as E_0^- itself, fixed by the duality automorphism, and therefore it must correspond to the orbit of a skew-map $W\mathcal{O}(-2) \to W^*\mathcal{O}(-1)$.

For $d = 5$ we have four walls (see [BMW14] for details): the walls produced by the destabilizing objects $\mathcal{O}(1)$ and $\mathcal{I}_p(1)$, W_Θ, and W_C. At the first two walls the Jordan-Holder filtrations have length 1, and so the strict transform of E_0^- before the divisorial contraction consists only of stable objects. As above, the divisorial contraction produces objects that are S-equivalent to complexes fitting into an exact sequence

$$0 \to \mathcal{O}(-3)[1] \to E \to \mathcal{O} \oplus \mathcal{O}_\ell(-3) \oplus \mathcal{O}_\ell(1) \to 0.$$

In $N(3,5,5)$ these correspond to the $GL(W) \times GL(W^*)$-orbits of matrices

$$\begin{pmatrix} 0 & -z & x & 0 & 0 \\ z & 0 & -y & 0 & 0 \\ -x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L \\ 0 & 0 & 0 & -L & 0 \end{pmatrix}$$

where L is a linear equation defining ℓ. The $GL(W) \times GL(W^*)$-orbits of these matrices are strictly semistable in $N(3,5,5)$. For $d > 5$ the orbits generated after the divisorial contraction are the orbits of elements of the form

$$\begin{pmatrix} 0 & -z & x & 0 & 0 \\ z & 0 & -y & 0 & 0 \\ -x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A \\ 0 & 0 & 0 & -A^t & 0 \end{pmatrix}$$

where A is a square matrix of order $(d-3)/2$, but we can not say much about it, only that when $\mathcal{G}$ is a sheaf then $\det A$ must give an equation for the curve C.

Now assume that B is a skew-symmetric matrix giving a $GL(W) \times GL(W^*)$-stable orbit. If there are invertible matrices $T, S \in GL(W)$ such that TBS^t is again skew-symmetric, then

$$B = (S^{-1}T)B(T^{-1}S)^t,$$

and therefore $S = \lambda T$ for some $\lambda \in \mathbb{C}^*$ since B is stable, and so $\mathrm{Hom(B, B)} = \mathbb{C}$.

Since $GL(W)$ can be embedded via the diagonal $T \mapsto (T, T^t)$ into $GL(W) \times GL(W^*)$ then a skew-symmetric matrix that is $GL(W) \times GL(W^*)$-stable is also $GL(W)$-stable for the diagonal action. Thus we can define an injective map

$$\{\text{Stable Skew } GL(W) \times GL(W^*) - \text{orbits}\} \longrightarrow \wedge^2 W \otimes V//GL(W),$$

where $V = \mathrm{Hom}(\mathcal{O}(-2), \mathcal{O}(-1))$. In the examples above, this map can be extended to the semistable orbits that have a *skew* representative. In fact, in a personal communication to the author, Aaron Bertram has made the following

Conjecture 4.17. *The last birational model of* $\mathrm{bl}_X E_0^-$ *is isomorphic to the GIT quotient* $\wedge^2 W \otimes V//GL(W)$.

4.3.3 Odd Veronese embeddings

One could as well study flips for secant varieties of odd Veronese embeddings by studying the Bridgeland wall crossing for the Gieseker moduli space containing curves of a fixed even degree. The invariants in this case are $v = (0, d, -d)$, and we want to run the MMP on $M_H(v) = N_{\mathbb{P}^2}(d, d/2)$.

The center of the walls is -1, and the radius of a wall produced by a subobject of invariants (r, c_1, c_2) is

$$R = \sqrt{1 + 2\frac{c_1 + c_2}{r}}.$$

Thus the category we have to consider is $\mathcal{A}_{-1}$. Again we obtain a collapsing wall W_C by noticing that every sheaf with these invariants will have positive Euler characteristic and therefore will have $\mathcal{O}$ as a subobject in the category $\mathcal{A}_{-1}$. Crossing this wall collapses an open set.

There is also a divisorial contraction produced by the tangent sheaf $T_{\mathbb{P}^2}(-1)$. To see this, notice that there is a rational map $N(d,d/2) \dashrightarrow N(2d,0)$ sending a semistable coherent sheaf $\mathcal{F}$ to the tensor product $\mathcal{F}\otimes\Omega^1(1)$. The moduli space $N(2d,0)$ has a natural Θ-divisor (sheaves with a section), and the image of $N(d,d/2)$ is not contained in Θ. By pulling back Θ we obtain a divisor Θ' consisting of semistable sheaves $\mathcal{F}$ such that $\mathcal{F}\otimes\Omega^1(1)$ has a section. Since

$$\mathrm{Hom}(\mathcal{O},\mathcal{F}\otimes\Omega^1(1)) = \mathrm{Hom}(T_{\mathbb{P}^2}(-1),\mathcal{F})$$

then the divisor Θ' is contracted when crossing the wall corresponding to the destabilizing object $T_{\mathbb{P}^2}(-1)$.

To analyze the last birational model one has to use the triad $\mathcal{O}(-1),\Omega^1(1),\mathcal{O}$ instead of $\mathcal{O}(-2),\mathcal{O}(-1),\mathcal{O}(-1)$ in the construction of the quiver moduli. This gives a construction of the last birational model as the GIT quotient

$$\mathrm{Hom}(\mathrm{V}\times\mathcal{O}(-2),\mathrm{W}\otimes\mathcal{O})//\mathrm{GL}(\mathrm{V})\times\mathrm{GL}(\mathrm{W}),$$

where V and W are complex vector spaces of dimension $d/2$.

From the inequalities in the previous chapter, one obtains that the invariants of a rank 1 destabilizing subobject producing a wall corresponding to a flip must satisfy

$$3 \le 2\chi - c_1 \le \frac{d^2}{4}+1 \ \text{ and } \ R \le c_1+1 \le d-R.$$

The exceptional locus for the first flip corresponds to those sheaves fitting into an exact sequence in $\mathcal{A}_{-1}$ of the form

$$0\to\mathcal{O}((d-2)/2)\to E\to\mathcal{O}((-d-2)/2)[1]\to 0.$$

The exceptional introduced when crossing this wall is

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}((d-2)/2),\mathcal{O}((-d-2)/2)[1])) = \mathbb{P}(H^0(\mathcal{O}(d-3)))^\vee.$$

All the results of the previous sections hold in this case, and we only have to notice that the correct duality automorphism is $(\cdot)^D\otimes\mathcal{O}(1)$ instead of $(\cdot)^D$.

We conclude this chapter with the following theorem, which is a corollary of our construction

Theorem 4.18. *Let $d \geq 5$ be an integer and let $\nu_{d-3} \colon \mathbb{P}^2 \to \mathbb{P}(H^0(\mathcal{O}(d-3)))^\vee = \mathbb{P}^N$ be $(d-3)$-uple embedding. There exists a sequence of flips*

$$\begin{tikzcd}
\mathrm{bl}_{\nu_{d-3}(\mathbb{P}^2)} \mathbb{P}^N \arrow[d] \arrow[dr] \arrow[rr, dotted, "f_1"] & & M_1 \arrow[dl] \arrow[dr] & \cdots & M_{\ulcorner \frac{d-1}{2} \urcorner} \arrow[dl] \\
N_1 \supset \mathbb{P}^N & M_1' & & \cdots &
\end{tikzcd}$$

where $Ex(f_i)$ is the strict transform of $Sec^i(\nu_{d-3}(\mathbb{P}^2))$ and N_1 is the first birational model of $N(d,0)$ or $N(d,d/2)$ depending on whether d is odd or even.

Remark 4.19. As we have seen, this sequence of flips is indeed longer but the exceptional loci after the first $\ulcorner (d-1)/2 \urcorner$ flips become more complicated since after this point strictly semistable objects have at least three Jordan-Holder factors.

CHAPTER 5

CHANGE OF POLARIZATION

The notion of stability for torsion-free sheaves on a smooth projective complex surface X depends on the choice of a divisor class $H \in \mathrm{Amp}(X)$ in the ample cone of the surface. As mentioned in Chapter 1, the coarse moduli spaces $M_H(v)$ of H-Gieseker semistable sheaves on X with Chern character v are projective and can be constructed via Geometric Invariant Theory (GIT) ([Gie77]). There is a wall and chamber decomposition of the ample cone of the surface $\mathrm{Amp}(X)$ such that $M_H(v)$ and $M_{H'}(v)$ are isomorphic when H and H' belong to the same chamber. In the 90s there was a great deal of interest in studying how these moduli spaces relate to each other for polarizations in different chambers. Results obtained independently by Ellingsrud and Göttsche [EG95] and Friedman and Qin [FQ95] for rank-two sheaves, and by Matsuki and Wentworth [MW97] in arbitrary positive rank, show that when crossing a wall in $\mathrm{Amp}(X)$, the moduli space $M_H(v)$ goes through a sequence of "Thaddeus flips" of moduli spaces of twisted sheaves.

Recall from Chapter 2 that if L is a $\mathbb{Q}$-line bundle on X, then a torsion-free sheaf E is L-twisted H-Gieseker semistable [MW97, Definition 3.2] (compare to Definition 2.17) if for all subsheaves $A \hookrightarrow E$ one has:

$$\left(\frac{\chi(A \otimes L)}{r(A)} - \frac{\chi(E \otimes L)}{r(E)}\right) + t(\mu_H(A) - \mu_H(E)) \leq 0 \;\text{ for }\; t \gg 0, \tag{*}$$

where the Euler characteristic $\chi(\_\_ \otimes L)$ is defined formally via the Riemann-Roch Theorem. Coarse moduli spaces of L-twisted H-Gieseker semistable sheaves were constructed in [MW97] and proven to be projective.

By Theorem 2.18, we know that moduli spaces of L-twisted H-Gieseker semistable sheaves are moduli spaces of Bridgeland semistable objects. Then one can ask if the variation of GIT obtained by Matsuki and Wentworth relating moduli spaces of Gieseker semistable sheaves for different polarizations can be interpreted as Bridgeland wall-crossings, and moreover if we can find a specific family of stability conditions realizing this variation.

For a fixed Chern character cohomology "vector" $v = (r(v), c_1(v), ch_2(v))$ on X, the vectors

$$\alpha_t = \left(1, -\frac{K_X}{2} + L + tH, d_t\right) \text{ with } d_t = -\frac{\chi(v)}{r(v)} - \frac{c_1(v)}{r(v)}(L + tH) + \chi(\mathcal{O})$$

are perpendicular to v in cohomology; i.e.,

$$\langle v, \alpha_t \rangle = \int_X ch_2(v) + c_1(v) \cdot \left(-\frac{K_X}{2} + L + tH\right) + r(v)d_t = 0.$$

If E is a torsion-free sheaf with $ch(E) = v$ and $A \hookrightarrow E$, then

$$\begin{aligned}
\langle ch(A), \alpha_t \rangle &= \chi(A) + c_1(A)(L + tH) + r(A)d_t - r(A)\chi(\mathcal{O}) \\
&= r(A)\left(\frac{\chi(A)}{r(A)} + \frac{c_1(A)}{r(A)}(L + tH) + d_t - \chi(\mathcal{O})\right) \\
&= r(A)\left(\frac{\chi(A \otimes L)}{r(A)} - \frac{\chi(E \otimes L)}{r(E)} + t(\mu_H(A) - \mu_H(E))\right).
\end{aligned}$$

Thus E is L-twisted H-Gieseker semistable if and only if for every subsheaf $A \hookrightarrow E$ one has

$$\langle ch(A), \alpha_t \rangle \leq 0 \text{ for } t \gg 0.$$

This allows us to study the change of polarization. The main result of this chapter is

Theorem (Theorem 5.7)**.** *Given H' and H'' two ample classes in adjacent chambers in the wall and chamber decomposition of $Amp(X)$ for the class v, then there is a one dimensional family of stability conditions $\{\sigma_t\}_{t \in (-1,1)}$ and rational numbers $-1 = t_0 < t_1 < \cdots < t_n = 1$ such that each moduli space $M_{\sigma_t}(v)$ of Bridgeland semistable objects is a moduli space of twisted sheaves for every t and is constant on (t_i, t_{i+1}), and it equals $M_{H'}(v)$ for $t \in (t_0, t_1)$ and equals $M_{H''}(v)$ for $t \in (t_{n-1}, t_n)$.*

Thus every wall in $\mathrm{Amp}(X)$ corresponds to a finite sequence of Bridgeland walls, and it will follow from the proof that we get a Bridgeland wall for each Thaddeus flip obtained in [MW97].

Remark 5.1. The results in this chapter were obtained in joint work with A. Bertram [BM15]. While preparing this manuscript the author was informed by Kota Yoshioka about his work on perverse sheaves [Yos14]. With a very different method he also obtains Theorem 5.7.

5.1 Notation

For a fixed Chern character $v = (r(v), c_1(v), ch_2(v))$ on X, the set of hyperplanes

$$\mathcal{H}_A = \{H \in \mathrm{Amp}(X)_{\mathbb{Q}} \colon \mu_H(A) = \mu_H(v)\},$$

where A supports an injective map $A \hookrightarrow E$ to some slope semistable sheaf E of type v and $c_1(A)/r(A) \neq c_1(v)/r(v)$ in $N^1(X)_{\mathbb{Q}}$, is locally finite. Moreover, if Δ is a finitely generated convex cone in $\mathrm{Amp}(X)_{\mathbb{Q}}$, then only finitely many of these hyperplanes intersect $\Delta \setminus \{0\}$ ([MW97]). We refer to these hyperplanes as walls for slope stability; every connected component of $\mathrm{Amp}(X)_{\mathbb{Q}} \setminus \bigcup_A \mathcal{H}_A$ is called a chamber. A wall $\mathcal{H}_A$ is a wall for Gieseker stability for the class $v = ch(E)$ if also

$$\frac{\chi(A)}{r(A)} - \frac{\chi(E)}{r(E)} \geq 0.$$

The wall and chamber decomposition for Gieseker stability is not well behaved since, in general, a torsion-free sheaf can pass from being stable to being unstable without ever being strictly semistable. For an example of this phenomena see Example 5.9.1.

Denote by $\widetilde{\mathrm{NS}}(X)$ the extended Neron-Severi group

$$H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z}).$$

As observed by A. Bertram in [Ber14], a vector $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \widetilde{\mathrm{NS}}(X) \otimes \mathbb{Q}$ with $\alpha_0 > 0$ and satisfying the Bogomolov inequality $\alpha_1^2 > 2\alpha_0\alpha_2$ induces a stability condition $\sigma_\alpha^H = (Z_\alpha^H, \mathcal{A}_\alpha^H) \in \mathrm{Stab}(X)$ for every ample class $H \in N^1(X)$. The heart $\mathcal{A}_\alpha^H$ is obtained in the usual way by tilting with respect to the torsion pair

$$\mathcal{Q}_\alpha^H = \left\{ Q \in \mathrm{Coh}(X) \colon \frac{1}{r(B)} \langle ch(B), \alpha H \rangle > 0 \ \text{ for all quotients } \ Q \twoheadrightarrow B \right\},$$
$$\mathcal{F}_\alpha^H = \left\{ F \in \mathrm{Coh}(X) \colon \frac{1}{r(A)} \langle ch(A), \alpha H \rangle \leq 0 \ \text{ for all subsheaves } \ A \hookrightarrow F \right\},$$

and the central charge is given by

$$Z_\alpha^H(E) = -\langle ch(E), \alpha \rangle + \sqrt{-1}\langle ch(E), \alpha H \rangle,$$

where αH is the product in cohomology

$$(\alpha_0 + \alpha_1 + \alpha_2)H = \alpha_0 H + \alpha_1 H = (0, \alpha_0 H, \alpha_1 H).$$

5.2 A boundedness result

Fix a Chern character vector $v = (c_0, c_1, c_2) \in \widetilde{NS}(X)_\mathbb{Q}$ with $c_0 \neq 0$, and an ample class $H \in N^1(X)_\mathbb{Q}$. Consider the vector

$$\alpha_t = (1, D_t, d_t),$$

where $D_t = -\frac{K_X}{2} + tH$ and $d_t = -\frac{1}{c_0}(c_1 D_t + c_2)$. With this choice of d_t we have $\langle v, \alpha_t \rangle = 0$. For an ample class $H' \in N^1(X)_\mathbb{Q}$ (not necessarily equal to H) and t such that $D_t^2 > 2d_t$ we obtain a stability condition $\sigma_t^{H'} := \sigma_{\alpha_t}^{H'}$. An object $E \in \mathcal{A}_t^{H'}$ with $ch(E) = v$ is $\sigma_t^{H'}$-semistable if and only if for all subobjects $A \hookrightarrow E$ in $\mathcal{A}_t^{H'}$ one has

$$\frac{\langle ch(A), \alpha_t \rangle}{\langle ch(A), \alpha_t H' \rangle} \leq \frac{\langle v, \alpha_t \rangle}{\langle v, \alpha_t H' \rangle} \quad \Leftrightarrow \quad \langle ch(A), \alpha_t \rangle \leq 0. \tag{5.1}$$

By taking a close look at the category $\mathcal{A}_t^{H'}$, we see that a sheaf $\mathcal{F}$ is in this category if and only if all its H'-semistable factors F_i satisfy

$$\mu_{H'}(F_i) > \frac{K_X H'}{2} - tHH'.$$

Then intuitively the categories $\mathcal{A}_t^{H'}$ approach $\mathrm{Coh}(X)$ as t approaches infinity. Since condition (5.1) is equivalent to the Gieseker condition for $t \gg 0$, then intuitively $\sigma_t^{H'}$-stability coincides with H-Gieseker stability for $t \gg 0$. This is a version of Bridgeland's large volume limit [Bri08, Proposition 14.2]. The proof in our coordinates appears in [Ber14], and we will sketch it below for convenience of the reader. It is remarkable that this limit result is independent of the class H'. This is precisely the advantage of using our coordinates.

If we plan to find a family of stability conditions realizing the change of polarization for the Gieseker moduli spaces, we should start by finding stability conditions whose only semistable objects are precisely the Gieseker semistable sheaves. We have the candidates to realize Gieseker semistability, but we should remove the asymptotic condition on t. This is equivalent to proving that the values of t, corresponding to intersections of walls for the class v in $\mathrm{Stab}(X)$ and the "ray" $\{\sigma_t^{H'}\}_t$, are bounded above. Assume that $(H')^2 = HH'$. We want to prove the following

Theorem 5.2. *Fix $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$. Consider the vector $\alpha_t = (1, L + D_t, d_t)$ where as before d_t is chosen such that $\langle v, \alpha_t \rangle = 0$. Assume that H and H' are in the same chamber for L-twisted H-Gieseker stability. Then there exists $t_0 > 0$ such that for all $t > t_0$, an object E of class v is $\sigma_t^{H'}$-stable precisely if it is an L-twisted H-Gieseker stable sheaf.*

If an object E of class v is $\sigma_t^{H'}$-semistable for $t \gg 0$, then E must be a sheaf since otherwise $\mathcal{H}^{-1}(E)[1]$ would destabilize E for large t. Since the stability condition (5.1) (for the class v) is equivalent to the L-twisted H-Gieseker condition, then every sheaf that is $\sigma_t^{H'}$-semistable for all $t \gg 0$ should be L-twisted H-Gieseker semistable. Conversely, if $E \in \mathcal{A}_t^{H'}$ is an L-twisted semistable sheaf with $ch(E) = v$, then no subobject $A \hookrightarrow E$ can destabilize E for $t \gg 0$. Indeed, if A is a subobject of E for $t \gg 0$, then A must be a subsheaf and therefore can not destabilize E since

$$\langle ch(A), \alpha_t \rangle = r(A) \left(\frac{\chi(A \otimes L)}{r(A)} - \frac{\chi(E \otimes L)}{r(E)} \right) + r(A) \left(\mu_H(A) - \mu_H(E) \right) t.$$

When $H = H'$, the existence of t_0 follows from a boundedness result of Maciocia.

The standard coordinates introduced in Chapter 2 for geometric Bridgeland stability conditions depend on two numerical classes $\beta, H \in N^1(X)_{\mathbb{Q}}$ with H ample. The stability conditions $\sigma_{\beta,tH}$ are given by the vectors

$$\alpha_{\beta,tH} = \left(1, -\beta, \frac{\beta^2}{2} - \frac{t^2H^2}{2}\right)$$

that clearly satisfy the Bogomolov inequality for every $t > 0$. This choice of vectors is natural from the Physics point of view since the corresponding central charge takes the form

$$\int_X e^{-\beta-\sqrt{-1}tH} v = -\langle v, \alpha_{\beta,tH} \rangle + \sqrt{-1} \langle v, \alpha_{\beta,tH} H \rangle.$$

Theorem 5.3 ([Mac12]). *Write $\beta = x_0 H + u_0 G$ for some divisor G with $G \cdot H = 0$ and $G^2 = -1$. For $x \in \mathbb{R}$ let $\beta_x = xH + u_0 G$. Then, in the plane*

$$\Pi_{u_0} = \{\sigma^H_{\alpha_{x,y}} : \alpha_{x,y} = (1, -\beta_x, \frac{1}{2}(\beta_x^2 - y^2 H^2)), \;\; x \in \mathbb{R},\; y > 0\} \subset \mathrm{Stab}(X)$$

the walls are semicircles of bounded center and radius.

In particular, if $-\beta = -\frac{K_X}{2} + L + tH$ and $H' = H$, then the ray $\{\sigma_t^{H'}\}_t$ embeds as a one-parameter family in one of these half planes.

We remark that the standard techniques used in [LQ11] to show Theorem 2.22 do not work for us since the categories $\mathcal{A}_t^{H'}$ are changing for every t, and therefore the set of objects destabilizing an L-twisted H-Gieseker semistable sheaf for some $t > t_0$ is not necessarily bounded. We plan to extend Maciocia's result to the case when H and H' are in the same chamber for slope stability. It will then follow that H and H' can be chosen in the same chamber for Gieseker stability. We will need the following lemma.

Lemma 5.4. *If an L-twisted H-Gieseker semistable sheaf E of class v is $\sigma_{t_0}^{H'}$-semistable for some t_0, then E is $\sigma_t^{H'}$-semistable for all $t \geq t_0$.*

Proof. First, notice that $E \in \mathcal{A}_t^{H'}$ for all $t \geq t_0$. If E is unstable for some $t > t_0$, then there is $t_1 \geq t_0$ such that E is $\sigma_t^{H'}$-semistable for all $t \in [t_0, t_1]$ and strictly semistable at t_1. Notice that if $0 \to A \to E \to B \to 0$ is a short exact sequence in $\mathcal{A}_{t_1}^{H'}$ with $\langle ch(A), \alpha_{t_1} \rangle = 0$ and $\langle ch(A), \alpha_t \rangle > 0$ for all $t > t_1$, then A can not be a subsheaf of E since in such case we would have

$$\langle ch(A), \alpha_t \rangle = r(A) \left(\chi(A \otimes L) - \frac{\chi(E \otimes L)}{r(E)} + (\mu_H(A) - \mu_H(E))\, t \right) \leq 0 \;\text{ for }\; t > t_1,$$

due to the L-twisted H-Gieseker semistability of E. In particular A can not have rank 1.

Assume that $\mathcal{H}^{-1}(B) \neq 0$, and let $t_2 > t_1$ such that the first H'-semistable factor F_1 of $\mathcal{H}^{-1}(B)$ has slope $\mu_{H'}(F_1) = \frac{K \cdot H'}{2} - t_2 (H')^2$. Then $\langle ch(F_1), \alpha_{t_2} \rangle < 0$ and therefore the quotient $B/F_1[1]$ in $\mathcal{A}_{t_2}^{H'}$ satisfies $\langle ch(B/F_1[1]), \alpha_{t_2} \rangle < 0$. Thus, if $\mathcal{K}$ denotes the kernel in $\mathcal{A}_{t_2}^{H'}$ of the map $E \to B/F_1[1]$ then $\langle ch(\mathcal{K}), \alpha_{t_2} \rangle > 0$. Since $r(\mathcal{K}) = r(A) - r(F_1)$ then by induction on the rank of destabilizing subobjects of E

for $t > t_0$, we know that $\mathcal{K}$ destabilizes E for all $t \leq t_2$ as long as $\mathcal{K}$ is a subobject of E in $\mathcal{A}_t^{H'}$, in particular $\mathcal{K}$ destabilizes E at t_1. Thus $\mathcal{H}^{-1}(B) = 0$, and therefore A is a subsheaf of E and can not destabilize E for any $t > t_1$. $\square$

As mentioned before, assume that H and H' are ample classes such that the set $\{sH + tH' \colon s, t > 0\}$ is contained in a chamber for slope stability for the class v. We can consider the two-dimensional family of stability conditions $\sigma_{s,t} := \sigma^{H'}_{\alpha_{s,t}}$ given by the vectors

$$\alpha_{s,t} = (1, -\frac{K}{2} + sH + tH', d_{s,t}),$$

where $d_{s,t}$ is chosen such that $\langle v, \alpha_{s,t} \rangle = 0$. To ease the notation we denote the heart $\mathcal{A}^{H'}_{\alpha_{s,t}}$ by $\mathcal{A}_{s,t}$. Since our goal is to study the change of polarization for the Gieseker moduli and $M_H(v) \cong M_H(v \cdot ch(A))$ for any line bundle A, then by twisting the class v if necessary we can assume that

$$\frac{K^2}{8} > \chi(\mathcal{O}) - \frac{\chi(v)}{r(v)}, \text{ and } \mu_{H'}(v) > \frac{KH'}{2}. \tag{5.2}$$

This assumption is harmless since for instance we could twist by nH for n sufficiently large and divisible. This guarantees that $\sigma_{s,t}$ is a stability condition for all $s, t \geq 0$ and that every H'-semistable object of class v is in the category $\mathcal{A}_{s,t}$ for all $s, t \geq 0$. Under these assumptions we can describe the walls in the two-dimensional slice $\{\sigma_{s,t}\}_{s,t \geq 0}$.

Corollary 5.5. *Every wall for the class v in the quadrant $\{\sigma_{s,t}\}_{s,t\geq 0}$ of stability conditions is a line of nonpositive slope, and it has slope zero (or infinity) if and only if H (or H') is on a wall in the wall and chamber decomposition of the ample cone for Gieseker stability with respect to the class v.*

Proof. We first describe the horizontal walls. Assume that there is an inclusion $0 \to A \to E \to B \to 0$ of objects in $\mathcal{A}_{s_0,t_0}$ with $\mu_H(A) = \mu_H(E)$. If $\mathcal{H}^{-1}(B) \neq 0$, then there is $t_1 > t_0$ such that $\mu_{H'}(F_1) = \frac{K_X H'}{2} - (s_0 + t_1)(H')^2$, where F_1 is the first H'-semistable factor of $\mathcal{H}^{-1}(B)$, then the sheaf quotient A/F_1 is a destabilizing subobject of E in $\mathcal{A}_{s_0,t_1}$ contradicting Lemma 5.4. Thus A must be a subsheaf of E, implying that H is on wall for slope semistability with respect to the class v.

Moreover, since such wall is given by $t = c > 0$ then A should destabilize E with respect H-Gieseker stability. If on the other hand

$$0 \to A \to E \to B \to 0 \tag{$\diamond$}$$

is a short exact sequence of sheaves destabilizing E with respect to H-Gieseker stability then for

$$t_0 = \left(\frac{\chi(A)}{r(A)} - \frac{\chi(E)}{r(E)}\right) / (\mu_{H'}(E) - \mu_{H'}(A))$$

and s large enough ($\diamond$) is a short exact sequence in $\mathcal{A}_{s,t_0}$ and so produces a horizontal wall in the quadrant $\{\sigma_{s,t}\}_{s,t\geq 0}$.

Assume that $\mathcal{W}$ is a wall with positive slope destabilizing a sheaf E of class v that is both H-stable and H'-stable. We can assume that E is destabilized in a small open set of $\mathcal{W}$ containing the point (s_0, t_0). By Lemma 5.4 E is also $\sigma_{s_0+\epsilon,t_0}$-semistable and therefore $\sigma_{s_0+\epsilon,t_0+\delta}$-semistable for all $\delta > 0$ contradicting that $\mathcal{W}$ is a wall destabilizing E near (s_0, t_0). The exact same argument shows that if E is H-stable and strictly H'-semistable, then the existence of a vertical wall will prevent the existence of any wall of positive slope destabilizing E. $\square$

Remark 5.6. We can consider the n-dimensional family of stability conditions $\sigma_{\mathfrak{a}}^H$ given by the vectors

$$\alpha_{\mathfrak{a}} = (1, -\frac{K_X}{2} + a_1 H_1 + \cdots + a_n H_n, d_{\mathfrak{a}})$$

where all the H_i's are in the closure of the same chamber with respect to Gieseker stability for the class v, $a_k \geq 0$ for all k, and H is an interior point of the cone $\{a_1 H_1 + \cdots + a_n H_n \colon a_k \geq 0\} \subset \mathrm{Amp}(X)_{\mathbb{Q}}$, and again we choose $d_{\mathfrak{a}}$ such that $\langle v, \alpha_{\mathfrak{a}} \rangle = 0$. Then Corollary 5.5 remains true: a wall destabilizing is a hyperplane that intersects each "axis" nonnegatively; it is parallel to the H_i-axis and lies in $\{\sigma_{\mathfrak{a}}\}_{a_k \geq 0}$ if and only if H_i lies on a wall with respect to Gieseker stability for the class v. This can be easily proven by induction on n.

With the assumptions in (5.2) we can now prove Theorem 5.2.

Proof of Theorem 5.2. We prove the case $L = 0$, but the proof works for any $\mathbb{Q}$-line bundle L. Because of our assumptions and Theorem 5.3 we know that the walls on the ray $(0,t)$ are finite; therefore there are only finitely many walls intersecting this ray (since at a given point there are only finitely many Chern characters responsible for a wall). All these walls are lines of negative slope, and therefore there exists a constant $S > 0$ such that the walls coming from the ray $(0,t)$ intersect the ray $(s,0)$ at some $0 < s < S$.

Let T be the upper bound for the walls on the ray $(0,t)$; i.e., for all $t > T$ the only $\sigma_{0,t}$-semistable objects of class v are H'-Gieseker semistable sheaves (and therefore H-Gieseker semistable). We will prove that the walls in the ray $(s,0)$ are bounded by $M = \max\{S,T\}$.

Assume that there is a destabilizing sequence

$$0 \to A \to E \to B \to 0 \tag{$\star$}$$

in $\mathcal{A}_{s_0,0}$, destabilizing an H-Gieseker semistable sheaf E of class v for some $s_0 > M$. The corresponding wall $\mathcal{W} = \mathcal{W}(A,E)$ in the (s,t)-plane is a line of negative slope. Notice that $\mathcal{A}_{s_0,0} = \mathcal{A}_{0,s_0}$, and therefore ($\star$) is also an exact sequence in $\mathcal{A}_{0,s_0}$. But since $s_0 > T$ we know that E is σ_{0,s_0}-stable and A can not destabilize E at this point. Thus we obtain

$$\left(\frac{\chi(A)}{r(A)} - \frac{\chi(E)}{r(E)}\right) + s_0(\mu_{H'}(A) - \mu_{H'}(E)) < 0,$$
$$\left(\frac{\chi(A)}{r(A)} - \frac{\chi(E)}{r(E)}\right) + s_0(\mu_H(A) - \mu_H(E)) = 0.$$

Therefore the slope of $\mathcal{W}$ is > -1 and $B \in \mathcal{A}_{s,t}$ for all $(s,t) \in \mathcal{W}$ ($s,t \geq 0$).

Let

$$0 \to A_n \to A \to F_n \to 0$$

be the last part of the Harder-Narasimhan filtration of A with respect to H'-stability. Since A can not destabilize E at $(0,t_{\mathcal{W}}) = \mathcal{W} \cap \{(0,t) \colon t \geq 0\}$ and $E, B \in \mathcal{A}_{0,t_{\mathcal{W}}}$, then A can not belong to this category. Thus, there is $(s_1,t_1) \in \mathcal{W}$ such that $\mu_{H'}(F_n) = \frac{K_X H'}{2} - (s_1+t_1)(H')^2$, and therefore $\langle ch(A_n), \alpha_{s,t}\rangle > 0$ for (s,t) near (s_1,t_1). This implies that the wall $\mathcal{W}' = \mathcal{W}(A_n,E)$ intersects $\mathcal{W}$. The slopes of $\mathcal{W}$ and $\mathcal{W}'$ must be the same; otherwise repeating the argument in the proof of Corollary 5.5 (moving right and then up) we will contradict that $\mathcal{W}$ is a wall. By finiteness of the Harder-Narasimhan filtration of A we conclude that $\mathcal{W}$ should extend to intersect the ray $(0, t)$, contradicting the choice of M. $\square$

5.3 Change of the polarization

We will extend Theorem 5.2 just a little, enough to prove the following

Theorem 5.7. *[MW97, Theorem 5.6] Let H^+, H^- be two ample divisors on adjacent chambers with respect to the Giesker wall and chamber decomposition of $Amp(X)_{\mathbb{Q}}$ for the class v. There is a one-parameter family of stability conditions $\{\sigma_t\}_{t\in I}$ and $t_0 < t_1 < \cdots < t_n$ in I such that the moduli spaces $M_{\sigma_t}(v)$ are isomorphic for all $t \in (t_i, t_{i+1})$. Moreover, each of these moduli spaces is isomorphic to a moduli space of twisted sheaves. For $t < t_0$, $M_{\sigma_t}(v)$ coincides with the moduli space of H^+-Gieseker semistable sheaves, and for $t > t_n$ it coincides with the moduli of H^--Gieseker semistable sheaves.*

Proof. Let H_0 be an ample class on a wall separating the chambers containing H^+ and H^-. It is enough to find a family of stability conditions reflecting the change of polarization from H^+ to H_0. Consider the two-dimensional family of stability conditions $\sigma_{s,t}^{H^+}$ given by the vectors $\alpha_{s,t} = (1, -\frac{K}{2} + sH_0 + tH^+, d_{s,t})$ where as before $d_{s,t}$ is chosen such that $\langle \alpha_{s,t}, v\rangle = 0$. By Corollary 5.5 we know that in the quadrant $\{\sigma_{s,t}^{H^+}\}_{s,t\geq 0}$ there are horizontal walls (finitely many since there are only finitely many Chern characters of subsheaves destabilizing a Gieseker semistable sheaf [MW97, Proposition 1.6]). The key point in the proof of Theorem 5.2 that allows us to get only finitely many walls is the absence of horizontal walls, so in the first quadrant above the first horizontal wall there are only finitely many walls and the same remains true between two consecutive horizontal walls. Then by choosing s_0 large enough we know that the walls intersecting the ray $\Lambda_{s_0} = \{\sigma_{s_0,t}^{H^+} : t \geq 0\}$ are all horizontal.

Note that if t is a positive rational number, then an object is $\sigma_{s_0,t}^{H^+}$-semistable if and only if it is a tH^+-twisted H_0-Gieseker semistable sheaf. In particular $\sigma_{s_0,t}^{H^+}$-semistable objects have projective moduli [MW97].

By Corollary 5.5 we know that a wall on the ray Λ_{s_0} is produced if there is an H_0-semistable sheaf of class v that is H^+-Gieseker semistable but that fails to be

H_0-Gieseker semistable; i.e., if there is an inclusion of torsion-free sheaves $A \hookrightarrow E$ with $ch(E) = v$ such that

$$\mu_{H_0}(A) = \mu_{H_0}(v),$$
$$\mu_{H^+}(A) < \mu_{H^+}(v), \text{ and}$$
$$\frac{\chi(A)}{r(A)} > \frac{\chi(v)}{r(v)}.$$

This inclusion will produce the wall

$$t = -\left(\frac{\chi(A)}{r(A)} - \frac{\chi(v)}{r(v)}\right) / \left(\mu_{H^+}(A) - \mu_{H^+}(v)\right),$$

which is rational. Then there are $t_0 < t_1 < \cdots < t_k$ rational numbers corresponding to walls on the ray Λ_{s_0} such that for $t > t_k$, $\sigma^{H^+}_{s_0,t}$-semistability coincides with H^+-Gieseker semistability. □

Remark 5.8. If we consider the stability conditions of Remark 5.6, then the proof of Theorem 5.7 guarantees the existence of a convex and polyhedral chamber $\mathcal{C} \subset \Delta = \{a_1H_1 + \cdots + a_nH_n : a_i \geq 0\}$ such that the determinant line bundle associated to a stability condition for a polarization in $\mathcal{C}$ is an ample line bundle on the Gieseker moduli.

Remark 5.9. Figure 5.1 shows the typical picture of walls for slope stability and our Bridgeland walls. The red line corresponds to the one-dimensional family of stability conditions in Theorem 5.7. The chambers $\mathcal{C}^{\pm}$ are slices of $\mathrm{Amp}(M_{\mathcal{C}^{\pm}}(v))$. Notice that $\mathcal{W}_0$ may or may not be a Bridgeland wall; it will be if and only if

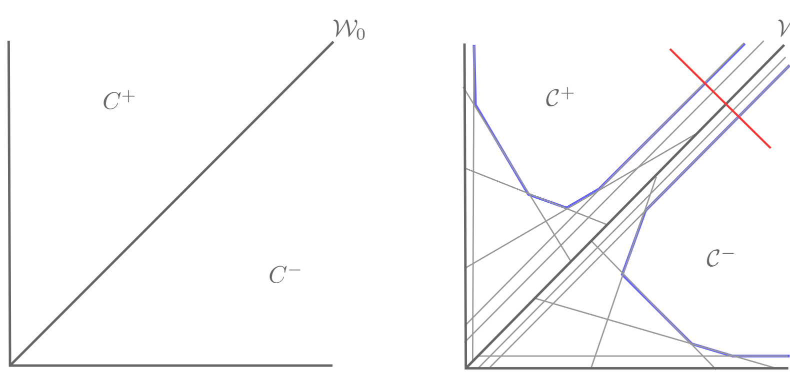

Figure 5.1. Gieseker and Bridgeland chambers.

there is an inclusion of sheaves $A \hookrightarrow E$ of the same reduced Hilbert polynomial and with E being C^+ or C^--stable of class v. If it happens that every H_0-Gieseker semistable sheaf is H_0-Gieseker stable for $H_0 \in \mathcal{W}_0$, then $\mathcal{W}_0$ is not a Bridgeland wall, and there are no Bridgeland walls parallel to $\mathcal{W}_0$; thus $\mathcal{C}^+$ and $\mathcal{C}^-$ give a single chamber. If every $C^{\pm}$-semistable sheaf is also H_0-Gieseker semistable and some $C^{\pm}$-semistable sheaf is strictly H_0-Gieseker semistable, then $\mathcal{W}_0$ is a Bridgeland wall, and there are no Bridgeland walls parallel to $\mathcal{W}_0$ on the side of $C^{\pm}$.

Example 5.9.1. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $v = (2, 0, -5)$. The advantage of studying rank-2 sheaves is that computing the walls for slope stability is very simple, but even in this case we can see some of the phenomena described in Remark 5.9 already happening. A wall for slope stability is produced by a short exact sequence

$$0 \to L \to E \to \mathcal{I}_Z(L^\vee) \to 0$$

for some line bundle L and a zero-dimensional subscheme $Z \subset X$ satisfying

$$L^2 + 5 = \ell(Z).$$

If $H_1 = \mathcal{O}(1,0)$ and $H_2 = \mathcal{O}(0,1)$, then L should also satisfy that for some integers $a, b \geq 0$

$$L \cdot (aH_1 + bH_2) = 0.$$

The ray generated by $aH_1 + bH_2$ is a wall. Thus $L = \pm(aH_1 - bH_2)$ and $L^2 = -2ab$, which implies $L^2 = -4, -2$, or 0. Therefore the walls for slope stability with respect to the class v are given by the polarizations $A_1 = H_1 + H_2$, $A_2 = 2H_1 + H_2$, $A_3 = H_1 + 2H_2$, H_1, and H_2. Since $\chi(E)/r(E) = -3/2$ and $\chi(L)$ is an integer then none of these walls is a Gieseker wall. However, each of the walls for slope stability produces a Bridgeland wall.

5.4 Birational geometry of complex surfaces

In this section we present a new approach to a result of Toda [Tod12] within the set of ideas surrounding the previous sections. The precise statement is

Theorem 5.10. *[Tod12, Corollary 1.4] Let X be a smooth projective complex surface, and let $\pi\colon X \to Y$ be the blow down of a -1-curve $C \subset X$. Then there is a continuous one* *parameter family of Bridgeland stability conditions* $\{\sigma_t\}_{t\in(-1,1)}$ *on* $D^bCoh(X)$ *such that* $M_{\sigma_t}([\mathcal{I}_p])$ *is isomorphic to* X *for* $t>0$ *and isomorphic to* Y *for* $t<0$.

Proof. Choose a sufficiently ample line bundle L on Y such that $\pi^*(L) = D = H+C$ for some $H \in \mathrm{Amp}(X)$. Consider the two-dimensional family of stability conditions $\sigma^H_{s,t}$ on X given by the vectors $\alpha_{s,t} = (1, -\frac{K_X}{2} + tH + sD, 1)$. By choosing s_0 large enough we can assume

$$-CH > \frac{K_X H}{2} - s_0 DH, \text{ and } \left(-\frac{K_X}{2} + tH + s_0 D\right)^2 > 2 \quad \text{for all} \quad t \geq 0.$$

Thus $0 \to \mathcal{O}(-C) \to \mathcal{I}_p \to \mathcal{O}_C(-1) \to 0$ is a short exact sequence in $\mathcal{A}^H_{s,t}$ for every $s > s_0$ and $t \geq 0$. Moreover,

$$\langle ch(\mathcal{O}(-C)), \alpha_{s,0}\rangle = -sDC = 0 = \langle ch(\mathcal{I}_p), \alpha_{s,0}\rangle.$$

Thus the ray $\sigma^H_{s,0}$ is a wall destabilizing all ideal sheaves $\mathcal{I}_p$ for $p \in C$. Now, if there were a horizontal wall in the quadrant $K = \{\sigma^H_{s,t} \colon s \geq s_0,\ t > 0\}$, then it would have to be produced by a subsheaf of $\mathcal{I}_p$, and such destabilizing object would have to be of the form $\mathcal{I}_Z(-C')$ for some zero-dimensional subscheme $Z \subset X$ containing p and some curve $C' \subset X$, but this is not possible since being the wall horizontal will force $C' = C$ and $\chi(\mathcal{I}_Z(-C)) - \chi(\mathcal{I}_p) = -\ell(Z) > 0$.

Therefore, by the proof of Theorem 5.2 we know that there are only finitely many walls in the closure $\bar{K}$. Thus, we can choose $s_1 > s_0$ such that there are no walls on the ray $\{\sigma^H_{s_1,t}\}_{t\geq 0}$ other than $\sigma^H_{s_0,0}$.

The moduli spaces $M_t([\mathcal{I}_p])$ for the stability conditions $\sigma^H_{s_1,t}$ with $t > 0$ are all isomorphic to the moduli space of s_1D-twisted H-semistable sheaves of class $[\mathcal{I}_p]$; i.e., they are all isomorphic to X. Since

$$\mathrm{ext}^1(\mathcal{O}(-C), \mathcal{O}_C(-1)) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 1,$$

then $M_t([\mathcal{I}_p])$ for $-1 \ll t \leq 0$ is naturally isomorphic to Y, and the map $M_t([\mathcal{I}_p]) \to M_0([\mathcal{I}_p])$ coincides with the contraction π. $\square$

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